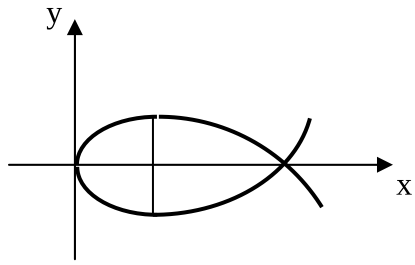
Oinarri Matematikoak I – Azterketa – 2. deialdia
Elektrizitate eta Elektronika 2004-09-06

LEHEN LAUHILABETE

A.1.- Lehen bi deribatu eta definizio-eremuaren azterketa analitikoa burutuz, egiaztatu irudikoa dela hurrengo funtzioaren adierazpide grafikoa:

$$9y^2 = x(x-3)^2$$



A.1.- OX simetria ardatzeko kurba bat da (funtzio bikoitza da)**:**

$$y = \pm\frac{1}{3}(x-3)\sqrt{x} = \mp\frac{1}{3}(3-x)\sqrt{x}$$

Definizio eremua: $x \geq 0 \quad \Rightarrow \quad D \equiv \{x \in R \,/\, x \geq 0\}$

Ardatzetako elkargune: $y = 0 \quad \Rightarrow \quad x = 0 \quad \wedge \quad x = 3 \quad \Rightarrow \quad A(0,0) \quad \wedge \quad B(3,0)$

1. Deribatu (y > 0) : $y = \frac{1}{3}(3-x)\sqrt{x} \quad \to \quad y' = \frac{1}{3}\left[-\sqrt{x} + \frac{3-x}{2\sqrt{x}}\right] = \frac{1-x}{2\sqrt{x}}$

Puntu singularrak:

y´= 0 $\Rightarrow$ 1-x = 0 $\Rightarrow$ x = 1 : C(1,-2/3)

y´= ∞ $\Rightarrow$ x = 0 : A(0,0)

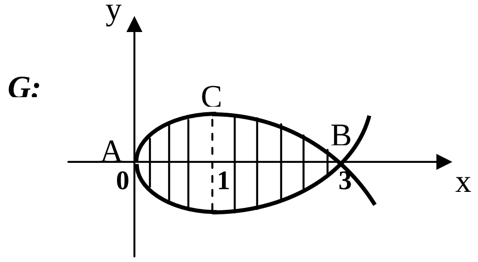
$y' > 0, \ \forall x \in (0,1)$ funtzio gorakorra, $y' < 0, \ \forall x > 1$ funtzio gorakorra:

$x = 1$: ukitzaile horizontaleko puntu: maximo erlatibo C(1, -2/3).

$x = 0$: ukitzaile bertikaleko puntu: A(0,0).

A.2.- Irudiko korapiloaren azalera kalkulatu.

Integralaren interpretazio geometrikoa da kalkuluaren oinarria:



Cálculo del área del lazo, A:

$$A = \frac{2}{3}\int_0^3 \sqrt{x}(3-x)\,dx = \frac{2}{3}\int_0^3 (3x^{1/2} - x^{3/2})\,dx = \frac{2}{3}\left[\frac{3x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right]\Bigg|_0^3 =$$

$$A = \frac{4}{3}\left[3\sqrt{3} - \frac{9}{5}\sqrt{3}\right] = \frac{8\sqrt{3}}{5} \quad (u^2)$$

A.3.- Froga ezazu $\mathrm{L} = \int_0^3 \frac{x+1}{\sqrt{x}}\,dx$ integralak irudiko korapiloaren luzeraren balioa adierazten duela. L kalkulatu.

Horrela da, zeren korapiloaren luzera, L, hurrengo eran kalkulatzen da:

$$\mathrm{L} = \int_0^3 \sqrt{1+(y')^2}\,dx = \int_0^3 \sqrt{1+\left(\frac{1-x}{2\sqrt{x}}\right)^2}\,dx = \int_0^3 \sqrt{\frac{x^2+2x+1}{4x}}\,dx = \int_0^3 \frac{x+1}{2\sqrt{x}}\,dx$$

Luzeraren kalkulua:

$$\mathrm{L} = \int_0^3 \frac{x+1}{\sqrt{x}}\,dx = \int_0^3 (\sqrt{x} + x^{-1/2})\,dx = \left[\frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2}\right]\Bigg|_0^3 = \left[\frac{2}{3}3\sqrt{3} + 2\sqrt{3} - 0\right] = 4\sqrt{3} \quad (u)$$

A.4.- Irudiko korapiloak OX ardatz inguru biratzean sortzen duen gainazalaren azalera kalkulatu.

$$A_{G-OX} = 2\pi\int_0^3 y\sqrt{1+(y')^2}\,dx = \frac{2\pi}{3}\int_0^3 \sqrt{x}(3-x)\frac{x+1}{\sqrt{x}}\,dx = \frac{2\pi}{3}\int_0^3 (3+2x-x^2)\,dx =$$

$$A = \frac{2\pi}{3}\left[3x + x^2 - \frac{x^3}{3}\right]\Bigg|_0^3 = \frac{2\pi}{3}\left[9+9-9\right] = 6\pi \quad (u^2)$$

B.1.- Biz hurrengo ekuazioaren bidez definitutako $z = z(x, y)$ funtzioa:

$$z^{-1} + \mathrm{Lny} = \Phi\left(\frac{y + x\mathrm{Lny}}{x}\right),$$

non (ϕ) funtzio arbitrarioa eta diferentziagarria den. Aurkitu ahalik eta era sinplifikatuenean hurrengo adierazpen diferentzialaren balioa: $E \equiv x(x+y)\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y}$

(x) eta (y) aldagaiekiko deribatuz, hurrenez hurren:

$$z^{-1} + \mathrm{Lny} = \Phi\left(\frac{y + x\mathrm{Lny}}{x}\right) = \Phi\left(\frac{y}{x} + \mathrm{Lny}\right)$$

$$\xrightarrow{\frac{\partial}{\partial x}} \quad -z^{-2}\cdot\frac{\partial z}{\partial x} = \Phi'\cdot\left(\frac{-y}{x^2}\right) \qquad \rightarrow \qquad \frac{\partial z}{\partial x} = \frac{y.z^2}{x^2}\cdot\Phi'$$

$$\xrightarrow{\frac{\partial}{\partial y}} \quad -z^{-2}\cdot\frac{\partial z}{\partial y} + \frac{1}{y} = \Phi'\cdot\left(\frac{1}{x} + \frac{1}{y}\right) \quad \rightarrow \quad \frac{\partial z}{\partial y} = \frac{z^2}{y} - \Phi'\cdot\frac{(x+y)z^2}{x\ y}$$

Aurreko deribatuak E adierazpen diferentzialean ordezkatuz:

$$E \equiv x(x+y)\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = x(x+y)\cdot\frac{yz^2}{x^2}\Phi' + y^2\cdot\left[\frac{z^2}{y} - \Phi'\cdot\frac{(x+y)z^2}{x\ y}\right] =$$

$$E \equiv (x+y)\frac{yz^2}{x}\cdot\Phi' + y^2\cdot\frac{z^2}{y} - (x+y)y^2\cdot\frac{z^2}{x\ y}\cdot\Phi' =$$

$$E \equiv (x+y)\frac{yz^2}{x}\cdot\left[\Phi' - \Phi'\right] + yz^2 = \boxed{yz^2}$$

B.2.- Askatu hurrengo hastapen baldintzatako ekuazio diferentziala:

$$y' = \frac{-yLny}{x - Lny}; \quad y(1) = e$$

$$(x - Lny)\frac{dy}{dx} = -yLny \quad \Leftrightarrow \quad \frac{dx}{dy} = \frac{x - Lny}{-yLny} = \frac{-1}{yLny}x + \frac{1}{y} \quad \Leftrightarrow \quad \boxed{x' + \frac{1}{yLny}x = \frac{1}{y}}$$

Ekuazio lineala, beraz (*zehatzgarria* baita ere. *Ikus:* μ = 1/y). Hurrengo eran askatuko dugu:

$$x = e^{-\int\frac{dy}{yL_n y}}\left[\int Lny\frac{1}{y}dy + C\right] = \frac{1}{Lny}\left[\frac{(Lny)^2}{2} + C\right] = \frac{Lny}{2} + \frac{C}{Lny}$$

edota: $\qquad 2x\cdot L_n y = (Lny)^2 + C$

Hastapen baldintza: $y(1) = e$;

$$2\cdot L_n e = (L_n e)^2 + C \quad \Leftrightarrow \quad 2 = 1 + C \quad \Leftrightarrow \quad C = 1$$

Soluzio partikularra, azkenik: $\qquad 2x\cdot L_n y = (Lny)^2 + 1$

BIGARREN LAUHILABETE

C.1.- Aska ezazu hurrengo hastapen baldintzatako ekuazio diferentziala:

$$x'' + 2x' + 5x = 3e^{-t}\cos 2t; \quad x(0) = x'(0) = 0$$

Laplace-ren $\mathcal{L}$ eragilea aplikatuz, *X(p)* transformatua askatuko da:

$$\left(p^2+2p+5\right)X(p) = \frac{3(p+1)}{(p+1)^2+4} \quad \rightarrow \quad X(p) = \frac{3(p+1)}{\left[(p+1)^2+4\right]^2}$$

Idatz daitekeena: $X(p) = \dfrac{p+1}{(p+1)^2+4}\cdot\dfrac{3}{(p+1)^2+4} = F(p)\cdot G(p)$

Konboluzio teoremarekin baliatuz, hurrengo integrala ebatziz lortuko da *X(p)* –ren alderantzizkoa:

$$x(t) = \mathcal{L}^{-1}[X(p)] = \mathcal{L}^{-1}[F(p)\cdot G(p)] = \int_0^t f(u)\cdot g(t-u)du$$

Aldez aurretik *f(x)* eta *g(x)* *Alderantzizko Taula-ren* bidez kalkulatuko ditugu:

$$f(x) = \mathfrak{L}^{-1}\left[\frac{p+1}{(p+1)^2+4}\right] = e^{-t}cos\,2t \quad ; \quad g(x) = \mathfrak{L}^{-1}\left[\frac{3}{(p+1)^2+4}\right] = \frac{3}{2}\,e^{-t}sin\,2t$$

Azkenik:

$$x(t) = \int_0^t e^{-u}cos\,2u\cdot\frac{3}{2}\,e^{-(t-u)}sin\,2(t-u)\,du = \frac{3}{2}e^{-t}\int_0^t cos\,2u\cdot sin\,2(t-u)\ du =$$

$$x(t) = \frac{3}{2}e^{-t}\int_0^t\left[cos\,2u\,(sin\,2t\cdot cos\,2u - cos\,2t\cdot sin\,2u)\ \right]du =$$

$$x(t) = \frac{3}{2}e^{-t}\left[\,sin\,2t\int_0^t cos^2\,2u\ \ du - cos\,2t\int_0^t\ sin\,2u\ cos\,2u\ du\right] =$$

$$x(t) = \frac{3}{2}e^{-t}\left[\frac{sin\,2t}{2}\left(t+\frac{sin\,2t\cdot cos\,2t}{2}\right) - cos\,2t\ \frac{sin^2\,2t}{2\cdot 2}\right] =$$

$$x(t) = \frac{3}{4}e^{-t}\left[\,t\,sin\,2t + \frac{sin^2\,2t\cdot cos\,2t}{2} - \frac{cos\,2t\cdot sin^2\,2t}{2}\right] = \frac{3}{4}t\,e^{-t}\ sin\,2t$$

C.2.- Kalkulu operazionala erabiliz, aska ezazu hurrengo ekuazio integrala:

$$x(t) = \frac{t^2}{2} - \int_0^t x(u)Sh(t-u)du$$

Laplace-ren $\mathcal{L}$ eragilea aplikatuz, *X(p)* transformatua askatuko da:

$$X(p)=\frac{1}{2}\frac{2}{p^3}-X(p)\cdot\frac{1}{p^2-1} \quad\rightarrow\quad X(p)\left[1+\frac{1}{p^2-1}\right]=\frac{1}{p^3} \quad\rightarrow\quad X(p)=\frac{p^2-1}{p^5}$$

$$X(p)=\frac{1}{p^3}-\frac{1}{p^5}$$

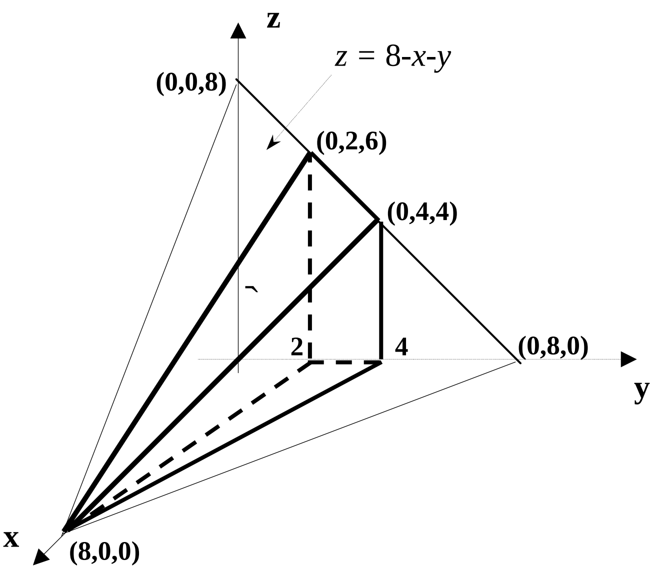
$\mathcal{L}^{-1}$ alderantzizko eragilearekin *x(t)* erdietsiko dugu:

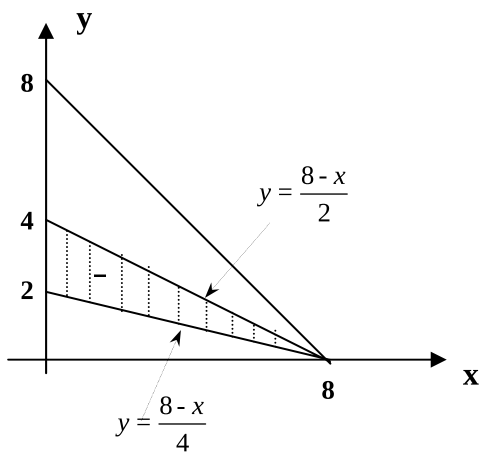
$$x(t)=\mathcal{L}^{-1}\left[X(p)\right]=\mathcal{L}^{-1}\left[\frac{1}{p^3}-\frac{1}{p^5}\right]=\frac{1}{2}t^2-\frac{1}{24}t^4$$

D.- Oktante positiboko [C] gorputz homogenoaren muga-gainazalak hurrengoak dira:

$$[\sigma_1]:\quad x+y+z-8=0;\quad [\sigma_2]:\quad x+4y-8=0\quad [\sigma_3]:\quad x+2y-8=0$$

D.1.- [C] gorputz homogenoaren grabitate-zentru geometrikoaren x_c koordenatua kalkulatu.





XOY plano gaineko [C] gorputzaren proiekzio zilindrikoa.

Grabitate-zentru geometrikoa. x_c koordenatuaren kalkulua (*homogenoa:δ konstantea)*.

$$x_c=\frac{M_{YOZ}}{m_{\mathbb{C}}}=\frac{\delta\iiint_V xdxdydz}{\delta\iiint_V dxdydz}=\frac{\iiint_V xdxdydz}{\iiint_V dxdydz}=\frac{\iiint_V xdxdydz}{V_C}$$

Bolumenaren kalkulu

$$V_C = \iiint_C dx\,dy\,dz = \iint_{D-XOY} dx\,dy \int_0^{8-x-y} dz = \int_0^8 dx \int_{(8-x)/4}^{(8-x)/2} (8-x-y)\,dy =$$

$$V_C = \int_0^8 \left[\frac{(8-x-y)^2}{-2} \right] \Bigg|_{(8-x)/4}^{(8-x)/2} dx = \frac{-1}{2} \int_0^8 \left[\left(\frac{8-x}{2} \right)^2 - \left(\frac{3(8-x)}{4} \right)^2 \right] dx =$$

$$V_C = \frac{-1}{2} \left[\left(\frac{1}{2} \right)^2 - \left(\frac{3}{4} \right)^2 \right] \int_0^8 (8-x)^2 \; dx = \frac{5}{32} \left[\frac{(8-x)^3}{-3} \right] \Bigg|_0^8 = \frac{5}{32} \cdot \frac{8^3}{3} = \frac{80}{3}$$

YOZ planoarekiko momentuaren kalkulua:

$$M_{YOZ} = \iiint_V xdxdydz = \iint_D xdxdy \int_0^{8-x-y} dz = \int_0^8 xdx \int_{(8-x)/4}^{(8-x)/2} (8-x-y)\,dy =$$

$$M_{YOZ} = \int_0^8 \left[\frac{(8-x-y)^2}{-2} \right] \Bigg|_{(8-x)/4}^{(8-x)/2} xdx = \frac{-1}{2} \int_0^8 \left[\left(\frac{8-x}{2} \right)^2 - \left(\frac{3(8-x)}{4} \right)^2 \right] xdx =$$

$$M_{YOZ} = \frac{-1}{2} \left[\left(\frac{1}{2} \right)^2 - \left(\frac{3}{4} \right)^2 \right] \int_0^8 x\left(8-x\right)^2 dx = \frac{5}{32} \int_0^8 \left(64x - 16x^2 + x^3\right) dx =$$

$$M_{YOZ} = \frac{5}{32} \left[32x^2 - \frac{16}{3} x^3 + \frac{x^4}{4} \right] \Bigg|_0^8 = 5\left[64 - \frac{256}{3} + 32 \right] = \frac{160}{3}$$

Azkenik: $$x_c = \frac{M_{YOZ}}{m_{\mathbb{C}}} = \frac{\iiint_V xdxdydz}{V_C} = \frac{160/3}{80/3} = 2\ (u)$$

D.2.- [C] gorputzaren $[\sigma_1]$ muga-gainazal atalaren azalera kalkulatu.

D.2.- $[\sigma_1]$ muga-gainazal atalaren azalera kalkulatu.

OZ-rekiko erregularra da $[\sigma_1]$ gainazal mugatzailearen atala, bere XOY plano gaineko proiekzioa aurreko D.1 –eko integral hirukoitzetako eremua den ¯ triangelu bera da. Integral bikoitza bilakatuz hurrengo eran kalkulatzen da Gainazal Integrala:

$$S_{\sigma_1} = \iint_{\sigma_1} d\sigma = \iint_D \frac{dxdy}{|\cos\gamma|}$$

$$*****[\sigma_1]: \quad x+y+z-8=0 \quad \to \quad \vec{n}_{\sigma_1} = \frac{[1,1,1]}{\sqrt{1^2+1^2+1^2}} \equiv \frac{[1,1,1]}{\sqrt{3}} \quad \to$$

$$|\cos\gamma| = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad S_{\sigma_1} = \iint_D \frac{dxdy}{|\cos\gamma|} = \iint_D \sqrt{3}\, dxdy = \sqrt{3}\int_0^8 dx \int_{(8-x)/4}^{(8-x)/2} dy =$$

$$S_{\sigma_1} = \sqrt{3}\left(\frac{1}{2}-\frac{1}{4}\right)\int_0^8 (8-x)dx = \frac{\sqrt{3}}{4}\left[\frac{(8\text{ - }x)^2}{\text{-}2}\right]\Bigg|_0^8 = \frac{\sqrt{3}\cdot 8^2}{8} = 8\sqrt{3} \quad (u^2)$$

MATEMATIKAREN HEDAPENA – AZTERKETA FINALA – BILBO, 03-06-09
Injeniaritza Teknikoa Elektrizitate eta Elektronikan

A ORRIA

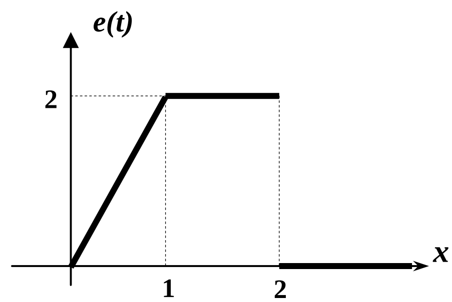
Ebatz itzazu ondorengo ekuazio diferentzialak:

[1] $y'' - 4y' + 4y = 12x^2e^{2x}$; $y(0) = 0,\ \ y'(0) = 1$

[2] $x'' + 9x = \dfrac{3}{sin3t}$

B ORRIA

[3] Aurki bedi $0 < x < 2$ tartean irudiko $e(t)$ funtzioarekin konbergentea den **Fourier-**en cosinuetako seriea.



$t = 0$ puntuko seriearen balio partikularraren bidez, dagokion zenbaki-seriearen balioa kalkulatu.

[4] LC serie-zirkuitu bati ondorengo eredu matematiko dagokio: $i'(t) + 10^4\int_0^t i(t)dt = e(t)$, non $e(t)$ [3] ariketan definituriko funtzioa den.

Askatu $i(t)$, $i(0)=0$ hastapen baldintzarako. $i(2)$ kalkulatu.

C ORRIA

[5] Biz D eremu planoa, ondorengo lerroek mugatutakoa,

$$x^2 + y^2 - 3 = 0 \;\; ; \quad x^2 + y^2 - 5 = 0 \;\; ; \quad x^2 - y^2 - 1 = 0 \;\; ; \quad x^2 - y^2 - 1 = 0$$

5.1. [D] eremua marraztu.

5.2 Adierazi ondorengo integralaren bi garapen eta limiteak (aurrenekoz *x-rekiko* integratuz, aurrenekoz *y-rekiko* integratuz, hurrenez-hurren):

$$\mathbf{I} = \iint_{\mathbf{D}} \mathbf{f(x,y)\,dx\,dy}$$

5.3 **f(x, y) = 8xy** denerako eta aldagai-aldaketa egokia erabiliz, **I** integrala kalkulatu.

[6] Biz **C** gorputz geometrikoa, ondorengo gainazalek mugatutakoa:

$$2(x^2 + y^2) - z^2 = 0 \;\; (0 \le z \le 4)\,; \quad x^2 + y^2 + 2z - 16 = 0 \;\; (4 \le z \le 6)\,; \; z = 0\,; \; z = 6$$

6.1 Aurki ezazu **C** gorputzaren bolumena.

6.2 Kalkulu integralen bidez aurkitu **C** gorputzaren azalera.

$y_1 = 4x^3$ *funtzioa ekuazio homogeno asoziatuaren soluzioa delarik, ondorengo ekuazio diferentzialaren soluzio orokorra kalkulatu:*

$$x(2x+3)\, y'' - 6(x+1)\, y' + 6y = 4x^3(2x+3)^2 \,,$$

Laplace-ren eragilea ondorengo sistemari aplikatuz, $\left\{ \begin{array}{l} x'(t) + y(t) = 0 \\ x(t) + y'(t) = \cos t \\ x(0) = 1 \,,\; y(0) = 0 \end{array} \right\} ,$

frogatu x(t) koordenatuaren ondorengo soluzioa: $x(t) = (Cht + \cos t)/2$

C.1 Aurki bedi $0 \le x \le 1$ *tartean* $y = 2 - x^2$ *parabolarantz konbergituko duen Fourier-en seriea.*

C.2 x=0 eta x=1 puntuetan aurkitu seriearen konbergentzia-balioak.

C.3 Fourier-en seriearen balio partikular baten bidez, egiaztatu ondorengo serie alternatuaren balioa:

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \ldots\ldots + (-1)^{n+1}\frac{1}{n^2} + \ldots\ldots = \frac{\pi^2}{12}$$

Marraztu ondorengo integral bikoitzaren $[D]$ *hedapen-eremua:*

$$I = \int_0^1 dy \int_{(y-2)^2}^{4y^2} f(x,y)\, dx + \int_1^2 dy \int_{(y-2)^2}^{4(2-y)} f(x,y)\, dx \,,$$

bere grabitate-zentru geometrikoaren koordenatuak kalkulatuz.

(E) *Biz* $[C]$ *gorputz geometrikoa, ondorengo gainazalek mugatutakoa:*

$$\begin{cases} x^2 + y^2 + z^2 - 8z = 0, & 0 \le z \le 1 \\ x^2 + y^2 + z - 8 = 0, & 1 \le z \le 7 \end{cases}$$

E.1 *Aurki ezazu* $[C]$ *gorputzaren grabitate-zentruaren kokapena.*

E.1 $[C]$ *gorputzaren azalera totala kalkulatu*

Oinarri Matematikoak I – Azterketa - Lehen partziala
Elektrizitate eta Elektronika 2004-05-31

LEHEN ATALA

A.1.- Marraz ezazu ondoko ekuazioa betetzen duten zenbaki konplexuen toki geometrikoa:

$$\overline{z+2}-4(z+2)^{-1}=0$$

(5 puntu)

A.2.- Gutxienezko azterketa analitikoarekin baliatuz, definizio-eremua eta deribatua bederen, marraz ezazu ondorengo ekuazioarekin definituriko funtzioaren adierazpen grafikoa:

$$y^2=\frac{1-x}{1+x}$$

(6 puntu)

A.3.- $I=2\int_0^1\sqrt{\frac{1-x}{1+x}}\,dx$ integralaren interpretazio geometrikoaren azalpena eman**.**

(2 puntu)

A.4.- Frogatu aurreko integrala $I=8\int_0^1\frac{t^2}{\left(1+t^2\right)^2}\,dt$ den, eta ordezkapen trigonometriko egoki baten bidez kalkulatu.

(7 puntu)

BIGARREN ATALA

B.1.- Biz $z=z(x,y)$ funtzioa, ondoko ekuazioaren bidez definitutakoa:

$$1+2\mathrm{xyz}=2\mathrm{x}^2\cdot\Phi\left(\frac{\mathrm{y}}{\mathrm{x}}\right)$$

non (ϕ) funtzio arbitrarioa eta diferentziagarria den.

Ahalik eta era simplifikatuenean hurrengo adierazpen diferentzialaren balioa aurkitu:
.

$$xy\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \right)$$ *(8 puntu)*

B.2.- Diferentzial zehatza bilakatuz, hurrengo ekuazio diferentziala integratu:

$$\left(2xy^2 + y\cos x\right)dx + \left(3x^2y + 2\sin x\right)dy = 0$$ *(6 puntu)*

B.3.- Ekuazio diferentzial linealera laburtuz, hurrengo ekuazio diferentziala integratu:

$$y' + y = (cosx - sinx)\,y^2$$

(6 puntu)

<u>JARRAIBIDEAK</u>

Era guztietako bibliografia erabilgarria da.

Azterketaren iraupena : **2 ordubete eta 45 minutu**.

Bananduta aurkeztuko dira **bi atalak**, ondorengo ordenean jasoko direlarik:
[1] : **azterketa hasi eta ordubete eta erdira.**
[2] : **2 ordubete eta 45 minutu** (*azterketaren amaieran*).

Azterketaren puntuaketa: **40 puntu, guztira** (20 *puntu atal bakoitzeko*).

Kalifikazioen publikazioa: Ekainaren 7an, eguerdiko 14etan. *(*5*n. Solairuko iragarki-oholean).*

Azterketa-Berrikuspena:
Laburategi Matematikoan *(*5*n. solairukoa)*, **ekainaren 9an, eguerdiko 12etan**.

Finala: HIRUGARREN ATALA

C.1.- Askatu ondorengo hastapen baldintzatako ekuazio diferentziala:

$$x'' + 2x' - 3x = 64te^{-3t}; \quad x(0) = x'(0) = 0$$

(5 puntu)

C.2.- $0 \le x \le 2$ tartean $y = -x^2 + 4x - 4$ parabolarekin konbergentea den Fourier-en cosinuetako serie bat aurkitu.
Kalkula ezazu seriearen balio partikularra *x = 0* puntuan.

(5 puntu)

2n Partziala: LEHEN ATALA

A.1.- Askatu ondorengo hastapen baldintzatako ekuazio diferentziala:

$$x'' + 2x' - 3x = 64te^{-3t}; \quad x(0) = x'(0) = 0$$ ***(7 puntu)***

A.2.- $0 \le x \le 2$ tartean $y = -x^2 + 4x - 4$ parabolarekin konbergentea den Fourier-en cosinuetako serie bat aurkitu.
Kalkula ezazu seriearen balio partikularra *x = 0* puntuan. ***(7 puntu)***

A.3.- Konboluzio-teoremarekin baliatuz, hurrengo hastapen-baldintzatako ekuazio diferentzialaren soluzioaren formula integrala kalkulatu

$$y'' - 4y' + 3y = f(t); \quad y(0) = 1, \; y'(0) = 1$$

(6 puntu)

C- A.1.- Askatu ondorengo hastapen baldintzatako ekuazio diferentziala:

$$x'' + 2x' - 3x = 64te^{-3t}; \quad x(0) = x'(0) = 0$$ ***(5 / 7 puntu)***

$$\mathfrak{L}\left[x'' + 2x' - 3x\right] = \mathfrak{L}\left[64te^{-3t}\right] \quad ; \quad x(0) = x'(0) = 0$$

$$p^2 \cdot X(p) + 2p \cdot X(p) - 3 \cdot X(p) = \frac{64}{(p+3)^2}$$

$$X(p) = \frac{64}{(p^2 + 2p - 3)(p+3)^2} = \frac{64}{(p-1)(p+3)(p+3)^2} = \frac{64}{(p-1)(p+3)^3} =$$

$$X(p) = \frac{64}{(p-1)(p+3)^3} = \frac{A}{p+3} + \frac{B}{(p+3)^2} + \frac{C}{(p+3)^3} + \frac{D}{p-1}$$

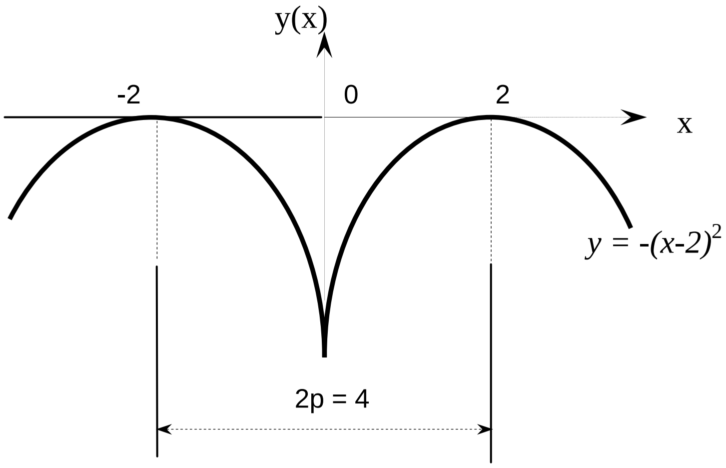
$$64 = A(p+3)^2(p-1) + B(p+3)(p-1) + C(p-1) + D(p+3)^3$$

3. maila:	**0 = A + D**	**A = - 1**
p=0:	**64 = -9A - 3B - C + 27D**	**B = - 4**
p= -3 :	**64 = -4 C**	**C = -16**
p= 1 :	**64 = 64 D**	**D = 1**

$$\boxed{\boldsymbol{x(t)}} = \mathfrak{L}^{-1}\left[\mathbf{X(p)}\right] = \mathfrak{L}^{-1}\left[\frac{-1}{p+3} + \frac{-4}{(p+3)^2} + \frac{-16}{(p+3)^3} + \frac{1}{p-2}\right] = \boxed{e^{2t} - e^{-3t}(1 + 4t + 8t^2)}$$

C-A.2.- $0 \le x \le 2$ tartean $y = -x^2 + 4x - 4$ parabolarekin konbergentea den Fourier-en cosinuetako serie bat aurkitu.
Kalkula ezazu seriearen balio partikularra $x = 0$ puntuan. ***(5 / 7 puntu)***

Se realiza una **prolongación par** como se muestra en la figura.



La serie y coeficientes de Fourier son:

$$a_0 = \frac{2}{p}\int_0^p f(t)dt; \quad a_k = \frac{2}{p}\int_0^p f(t)\cos\frac{k\pi t}{p}dt; \quad b_k = 0; \quad S(t) = \frac{a_0}{2} + \sum_1^{\infty} a_k \cos\frac{k\pi t}{p}$$

Al sustituir p = 2 y operar:

$$a_0 = -\int_0^2 (x-2)^2 dx = \frac{-1}{3}\left[(x-2)^3\right]\Big|_0^2 = \frac{-1}{3}\left[0-(-8)\right] = \boxed{\frac{-8}{3}}$$

$$a_k = -\int_0^2 (x-2)^2 \cos\frac{k\pi t}{2} dt$$

a_k zatika ebatziko da :

$$a_k = -\int_0^2 (x-2)^2 \cos\frac{k\pi t}{2} dx = \left| \begin{matrix} -(x-2)^2 = u & du = -2(x-2)dx \\ \cos\frac{k\pi x}{2}dx = dv & v = \frac{2}{k\pi}\sin\frac{k\pi x}{2} \end{matrix} \right| =$$

$$a_k = \left| -\frac{2(x-2)^2}{k\pi}\sin\frac{k\pi x}{2} \right|_0^2 + \underbrace{\frac{4}{k\pi}\int_0^2 (x-2)\sin\frac{k\pi x}{2}dx}_{I_1}$$

$$a_k = \left[0-(-8)\sin k\pi\right] + I_1 = I_1 = \frac{4}{k\pi}\int_0^2 (x-2)\sin\frac{k\pi x}{2}dx$$

$$a_k = \frac{4}{k\pi}\int_0^2 (x-2)\sin\frac{k\pi x}{2}\,dx = \left| \begin{array}{ll} (x-2)=u & du = dx \\ \sin\frac{k\pi x}{2}dx = dv & v = \frac{-2}{k\pi}\cos\frac{k\pi x}{2} \end{array} \right| =$$

$$a_k = \frac{4}{k\pi}\left[-\frac{2}{k\pi}(x-2)\cos\frac{k\pi x}{2}\right]\Bigg|_0^2 + \frac{8}{(k\pi)^2}\int_0^2 \cos\frac{k\pi x}{2}\,dx =$$

$$a_k = \frac{-8}{(k\pi)^2}\left[0-(-2\cos 0)\right] + \frac{8}{(k\pi)^2}\cdot\frac{2}{k\pi}\left[\sin\frac{k\pi x}{2}\right]\Bigg|_0^2 = \frac{-16}{(k\pi)^2} = a_k$$

$$\boxed{a_k = \frac{-16}{(k\pi)^2}} \qquad \boxed{a_0 = \frac{-8}{3}} \qquad \boxed{S(t) = \frac{a_0}{2} + \sum_1^{\infty} a_k \cos\frac{k\pi}{p}x}$$

Resulta la serie de cosenos:

$$S(t) = \frac{-4}{3} - \frac{16}{\pi^2}\sum_1^{\infty}\frac{1}{k^2}\cos\frac{k\pi}{2}x$$

En x = 0 la función periódica desarrollada presenta un punto de continuidad. De acuerdo con el teorema de Dirichlet se cumple:

$$S(0) = f(0) = -4 = \frac{-4}{3} - \frac{16}{\pi^2}\sum_1^{\infty}\frac{\cos 0}{k^2}$$

$$-4 = \frac{-4}{3} - \frac{16}{\pi^2}\sum_1^{\infty}\frac{1}{k^2} \rightarrow \qquad \sum_1^{\infty}\frac{1}{k^2} = S(k) = \left(-4+\frac{4}{3}\right)\left(\frac{-\pi^2}{16}\right) = \boxed{\frac{\pi^2}{6}}$$

Los primeros términos dela serie numérica obtenida son:

$$S(k) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{6}$$

A.3.- Konboluzio-teoremarekin baliatuz, hurrengo hastapen-baldintzatako ekuazio diferentzialaren soluzioaren formula integrala kalkulatu

$$y'' - 4y' + 3y = f(t); \quad y(0) = 1,\ y'(0) = 1$$

(6 puntu)

$$\mathfrak{L}\left[y'' - 4y' + 3y\right] = \mathfrak{L}\left[f(t)\right] \quad ; \quad y(0) = 1; \ \ y'(0) = 1$$

$$p^2 \cdot Y(p) - p - 1 - 4[p \cdot Y(p) - 1] + 3 \cdot Y(p) = F(p)$$

$$Y(p) = \frac{F(p)}{p^2 - 4p + 3} + \frac{p-3}{p^2 - 4p + 3} = \frac{F(p)}{(p-3)(p-1)} + \frac{p-3}{(p-3)(p-1)} = \frac{F(p)}{(p-3)(p-1)} + \frac{1}{p-1}$$

$$\boldsymbol{y(t)} = \mathfrak{L}^{-1}\left[Y(p)\right] = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p-3)(p-1)} + \frac{1}{p-1}\right] = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p-3)(p-1)}\right] + \mathfrak{L}^{-1}\left[\frac{1}{p-1}\right] =$$

$$\boldsymbol{y(t)} = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p-3)(p-1)}\right] + \boldsymbol{e^t} = \boldsymbol{e^t} + h(t) \qquad \textbf{(*)}$$

Konboluzio teorema

$$\boldsymbol{h(t)} = \mathfrak{L}^{-1}\left[F(p) \cdot \frac{1}{(p-3)(p-1)}\right] = \int_0^t \boldsymbol{f}(t-\tau) \cdot \boldsymbol{g}(\tau)\, \boldsymbol{d}\tau \quad , \ non \quad g(t) = \mathfrak{L}^{-1}\left[\frac{1}{(p-3)(p-1)}\right]$$

$$\mathbf{g(t)} = \mathfrak{L}^{-1}\left[\frac{1}{(p-3)(p-1)}\right] = \mathfrak{L}^{-1}\left[\frac{A}{p-3} + \frac{B}{p-1}\right] = \ldots.. = \mathfrak{L}^{-1}\left[\frac{1/2}{p-3} - \frac{1/2}{p-1}\right] = \frac{1}{2}\left[\mathbf{e^{3t}} - \mathbf{e^{t}}\right]$$

beraz:

$$\boldsymbol{h(t)} = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p-3)(p-1)}\right] = \frac{1}{2}\int_0^t \boldsymbol{f}(t-\tau) \cdot (e^{3\tau} - e^{\tau})\, \boldsymbol{d}\tau$$

azkenik: ***(*)***

$$\boxed{\boldsymbol{y(t)} = \boldsymbol{e^t} + \frac{1}{2}\int_0^t \boldsymbol{f}(t-\tau) \cdot (e^{3\tau} - e^{\tau})\, \boldsymbol{d}\tau}$$

C. A.1 $x'' + 2x' - 3x = 64te^{-3t};\quad x(0) = x'(0) = 0$ ***(5 / 7 puntu)***

Ekuazio karakteristikoa: $r^2 + 2r - 3 = 0 \;\Rightarrow\; r = 1\ (s = 1) \wedge r = -3\ (s = 1)$; $\text{SOS} = \left\{e^t, e^{-3t}\right\}$

Homogeno asoziatuaren Soluzio orokorra: $\boxed{x_{HA} = C_1 e^t + C_2 e^{-3t}}$

Parametroen Aldakuntza Metodoa

Ekuazio Osotuaren Soluzio Orokorra: $x_{HA} = C_1 e^t + C_2 e^{-3t} \Rightarrow \boxed{x_{OSO} = L_1 e^t + L_2 e^{-3t}}$ **(*)**

$$\begin{cases} \mathbf{L_1' e^t + L_2' e^{-3t} = 0} \\ \mathbf{L_1'(e^t) + L_2'(-3e^{-3t}) = 64te^{-3t}} \end{cases} \Rightarrow \begin{cases} \mathbf{L_1' + L_2' e^{-4t} = 0} \\ \mathbf{L_1' + L_2'(-3e^{-4t}) = 64te^{-4t}} \end{cases}$$

Ekuazioetako kenketatik: $L_2'(4 L_1' e^{-4t}) = -64e^{-4t} \Rightarrow L_2' = -16t$

$$L_1' = -L_2' e^{-4t} = 16te^{-4t}$$

$$\boxed{\begin{aligned} \mathbf{L_1} &= 16\int te^{-4t}dt = \left| \begin{matrix} u = t & du = dt \\ dv = e^{-4t}dt & v = \frac{-1}{4}e^{-4t} \end{matrix} \right| = 16\left[\frac{-1}{4}te^{-4t} + \frac{1}{4}\int e^{-4t}dt\right] = \\ \mathbf{L_1} &= 4\left[-te^{-4t} + \int e^{-4t}dt\right] = 4\left[-te^{-4t} + \frac{-1}{4}e^{-4t}\right] + \mathbf{C_1} = \left[-4t - 1\right]e^{-4t} + \mathbf{C_1} \end{aligned}}$$

$$\boxed{\mathbf{L_2} = -16\int t\,dt = -8t^2 + \mathbf{C_2}}$$

Ekuazio Osotuaren Soluzio Orokorra:

(*) $$x_{OSO} = e^t\left[\left[-4t-1\right]e^{-4t} + \mathbf{C_1}\right] + e^{-3t}\left[-8t^2 + \mathbf{C_2}\right] = \mathbf{C_1} e^t + \mathbf{C_2} e^{-3t} + e^{-3t}\left[-8t^2 - 4t - 1\right]$$

Azkenik: $$\boxed{x_{OSO} = \mathbf{C_1} e^t + \mathbf{C_2} e^{-3t} + e^{-3t}\left[-8t^2 - 4t - 1\right]}$$

Hastapen baldintzak

$$\begin{cases} \boxed{x = \mathbf{C_1} e^t + \mathbf{C_2} e^{-3t} + e^{-3t}(-8t^2 - 4t - 1)} \\ \boxed{x' = \mathbf{C_1} e^t - 3\mathbf{C_2} e^{-3t} + e^{-3t}\left[-3(-8t^2 - 4t - 1) + (-16t - 4)\right]} \end{cases} ; \begin{cases} x(0) = \mathbf{C_1} + \mathbf{C_2} - 1 = 0 \\ x'(0) = \mathbf{C_1} - 3\mathbf{C_2} - 1 = 0 \end{cases} ; \boxed{\begin{matrix} \mathbf{C_1} = 1 \\ \mathbf{C_2} = 0 \end{matrix}}$$

Soluzio Partikularra: $$\boxed{x_{OSO} = e^t + e^{-3t}\left[-8t^2 - 4t - 1\right]}$$

C. A.2 $x'' + 2x' - 3x = 64te^{-3t};\quad x(0) = x'(0) = 0$ ***(5 / 7 puntu)***

Ekuazio karakteristikoa: $r^2 + 2r - 3 = 0 \;\Rightarrow\; r = 1\ (s = 1) \;\wedge\; r = -3\ (s = 1)$; $\quad \text{SOS} = \left\{e^{t}, e^{-3t}\right\}$

Homogeno asoziatuaren Soluzio orokorra: $x_{HA} = C_1 e^{t} + C_2 e^{-3t}$

Koefiziente indeterminatuen metodoa:

$$\boxed{e^{-3t} \in x_{HA} \;/\; r = -3\ (s = 1)}$$

(*)

$$\underbrace{64te^{-3t}}_{gai\ ez-homogenoa} \Rightarrow X_{OSP} = \left(At + B\right)e^{-3t} \cdot t = \left(At^2 + Bt\right)e^{-3t}$$

$$X' = e^{-3t}\left[\,(2At + B) - 3(At^2 + Bt)\right] = e^{-3t}\left[\,(-3A)t^2 + (-3B + 2A)t + B\right]$$

$$X'' = e^{-3t}\left[\,9At^2 - 3(-3B + 2A)t - 3B - 6At + (-3B + 2A)\right] =$$
$$X'' = e^{-3t}\left[\,9At^2 + (9B - 12A)t + (-6B + 2A)\right]$$

$$X'' = e^{-3t}\left[\,9At^2 + (9B - 12A)t + (-6B + 2A)\right]$$
$$2X' = e^{-3t}\left[\,-6At^2 + (-6B + 4A)t + 2B\right]$$
$$-3X = e^{-3t}\left[-3At^2 - 3Bt\right]$$

$$X'' + 2X' - 3X = e^{-3t}\left[\,(0)t^2 - (8A)t + (-4B + 2A)\right] = 64te^{-3t}$$

Azkenik: $\left[-(8A)t + (-4B + 2A)\right] = 64t \;\Rightarrow\; \begin{cases} -8A = 64 \\ -4B + 2A = 0 \end{cases} \;\Rightarrow\; \boxed{\begin{matrix} A = -8 \\ B = -4 \end{matrix}}$

Ekuazio osotuaren Soluzio Partikular:

(*) $\quad \boxed{X = -(8t^2 + 4t)e^{-3t}}$

Ekuazio osotuaren Soluzio orokorra:

$$\boxed{x_{OSO}} = x_{HA} + X_{OSP} = \boxed{C_1 e^{t} + C_2 e^{-3t} - (8t^2 + 4t)e^{-3t}} \qquad (*)$$

Hastapen baldintzak

$$\begin{cases} \boxed{x = \mathbf{C_1} e^{t} + \mathbf{C_2} e^{-3t} - (8t^2 + 4t)e^{-3t}} \\ \boxed{x' = \mathbf{C_1} e^{t} - 3\mathbf{C_2} e^{-3t} + e^{-3t}\left[-3(-(8t^2 + 4t) + (-16t - 4)\right]} \end{cases} ; \quad \begin{cases} x(0) = \mathbf{C_1} + \mathbf{C_2} = 0 \\ x'(0) = \mathbf{C_1} - 3\mathbf{C_2} - 4 = 0 \end{cases} ; \quad \boxed{\begin{matrix} \mathbf{C_1} = 1 \\ \mathbf{C_2} = -1 \end{matrix}}$$

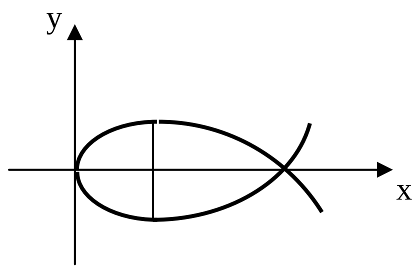
Soluzio Partikularra: (*) $\quad \boxed{x_{OSO} = e^{t} + e^{-3t}\left[-8t^2 - 4t - 1\right]}$

Oinarri Matematikoak I – Azterketa – 2. deialdia
Elektrizitate eta Elektronika 2004-09-06

LEHEN LAUHILABETE

A.1.- Lehen bi deribatu eta definizio-eremuaren azterketa analitikoa burutuz, egiaztatu irudikoa dela hurrengo funtzioaren adierazpide grafikoa:

$$9y^2 = x(x-3)^2$$



A.1.- OX simetria ardatzeko kurba bat da (funtzio bikoitza da)**:**

$$y = \pm\frac{1}{3}(x-3)\sqrt{x} = \mp\frac{1}{3}(3-x)\sqrt{x}$$

Definizio eremua: $x \geq 0 \quad \Rightarrow \quad D \equiv \{x \in R \,/\, x \geq 0\}$

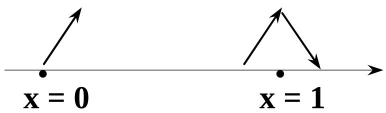
Ardatzetako elkargune: $y = 0 \quad \Rightarrow \quad x = 0 \quad \wedge \quad x = 3 \quad \Rightarrow \quad A(0,0) \quad \wedge \quad B(3,0)$

1. Deribatu (y > 0) : $y = \frac{1}{3}(3-x)\sqrt{x} \quad \rightarrow \quad y' = \frac{1}{3}\left[-\sqrt{x} + \frac{3-x}{2\sqrt{x}}\right] = \frac{1-x}{2\sqrt{x}}$

Puntu singularrak:

y´= 0 $\Rightarrow$ 1-x = 0 $\Rightarrow$ x = 1 : C(1,-2/3)

y´= ∞ $\Rightarrow$ x = 0 : A(0,0)



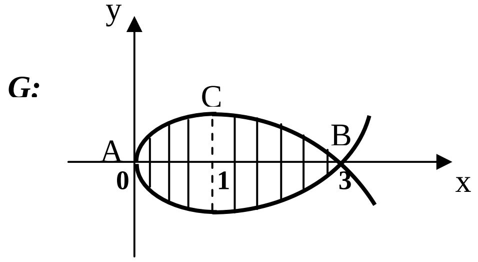
$y' > 0, \ \forall x \in (0,1)$ funtzio gorakorra, $y' < 0, \ \forall x > 1$ funtzio gorakorra:

$x = 1$: ukitzaile horizontaleko puntu: maximo erlatibo C(1, -2/3).

$x = 0$: ukitzaile bertikaleko puntu: A(0,0).

A.2.- Irudiko korapiloaren azalera kalkulatu.

Integralaren interpretazio geometrikoa da kalkuluaren oinarria:



Cálculo del área del lazo, A:

$$A = \frac{2}{3}\int_0^3 \sqrt{x}(3-x)\,dx = \frac{2}{3}\int_0^3 (3x^{1/2} - x^{3/2})\,dx = \frac{2}{3}\left[\frac{3x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right]\Bigg|_0^3 =$$

$$A = \frac{4}{3}\left[3\sqrt{3} - \frac{9}{5}\sqrt{3}\right] = \frac{8\sqrt{3}}{5} \quad (u^2)$$

A.3.- Froga ezazu $\mathrm{L} = \displaystyle\int_0^3 \frac{x+1}{\sqrt{x}}\,dx$ integralak irudiko korapiloaren luzeraren balioa adierazten duela. L kalkulatu.

Horrela da, zeren korapiloaren luzera, L, hurrengo eran kalkulatzen da:

$$\mathrm{L} = \int_0^3 \sqrt{1+(y')^2}\,dx = \int_0^3 \sqrt{1+\left(\frac{1-x}{2\sqrt{x}}\right)^2}\,dx = \int_0^3 \sqrt{\frac{x^2+2x+1}{4x}}\,dx = \int_0^3 \frac{x+1}{2\sqrt{x}}\,dx$$

Luzeraren kalkulua:

$$\mathrm{L} = \int_0^3 \frac{x+1}{\sqrt{x}}\,dx = \int_0^3 (\sqrt{x} + x^{-1/2})\,dx = \left[\frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2}\right]\Bigg|_0^3 = \left[\frac{2}{3}3\sqrt{3} + 2\sqrt{3} - 0\right] = 4\sqrt{3} \quad (u)$$

A.4.- Irudiko korapiloak OX ardatz inguru biratzean sortzen duen gainazalaren azalera kalkulatu.

$$A_{G-OX} = 2\pi\int_0^3 y\sqrt{1+(y')^2}\,dx = \frac{2\pi}{3}\int_0^3 \sqrt{x}(3-x)\frac{x+1}{\sqrt{x}}\,dx = \frac{2\pi}{3}\int_0^3 (3+2x-x^2)\,dx =$$

$$A = \frac{2\pi}{3}\left[3x + x^2 - \frac{x^3}{3}\right]\Bigg|_0^3 = \frac{2\pi}{3}[9+9-9] = 6\pi \quad (u^2)$$

B.1.- Biz hurrengo ekuazioaren bidez definitutako $z = z(x, y)$ funtzioa:

$$z^{-1} + \text{Lny} = \Phi\left(\frac{y + x\text{Lny}}{x}\right),$$

non (ϕ) funtzio arbitrarioa eta diferentziagarria den. Aurkitu ahalik eta era sinplifikatuenean hurrengo adierazpen diferentzialaren balioa: $E \equiv x(x+y)\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y}$

(x) eta (y) aldagaiekiko deribatuz, hurrenez hurren:

$$z^{-1} + \text{Lny} = \Phi\left(\frac{y + x\text{Lny}}{x}\right) = \Phi\left(\frac{y}{x} + \text{Lny}\right)$$

$$\xrightarrow{\frac{\partial}{\partial x}} \quad -z^{-2}\cdot\frac{\partial z}{\partial x} = \Phi'\cdot\left(\frac{-y}{x^2}\right) \qquad \to \qquad \frac{\partial z}{\partial x} = \frac{y.z^2}{x^2}\cdot\Phi'$$

$$\xrightarrow{\frac{\partial}{\partial y}} \quad -z^{-2}\cdot\frac{\partial z}{\partial y} + \frac{1}{y} = \Phi'\cdot\left(\frac{1}{x} + \frac{1}{y}\right) \quad \to \quad \frac{\partial z}{\partial y} = \frac{z^2}{y} - \Phi'\cdot\frac{(x+y)z^2}{x\ y}$$

Aurreko deribatuak E adierazpen diferentzialean ordezkatuz:

$$E \equiv x(x+y)\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = x(x+y)\cdot\frac{yz^2}{x^2}\Phi' + y^2\cdot\left[\frac{z^2}{y} - \Phi'\cdot\frac{(x+y)z^2}{x\ y}\right] =$$

$$E \equiv (x+y)\frac{yz^2}{x}\cdot\Phi' + y^2\cdot\frac{z^2}{y} - (x+y)y^2\cdot\frac{z^2}{x\ y}\cdot\Phi' =$$

$$E \equiv (x+y)\frac{yz^2}{x}\cdot\left[\Phi' - \Phi'\right] + yz^2 = \boxed{yz^2}$$

B.2.- Askatu hurrengo hastapen baldintzatako ekuazio diferentziala:

$$y' = \frac{-yLny}{x - Lny}; \quad y(1) = e$$

$$(x - Lny)\frac{dy}{dx} = -yLny \quad \Leftrightarrow \quad \frac{dx}{dy} = \frac{x - Lny}{-yLny} = \frac{-1}{yLny}x + \frac{1}{y} \quad \Leftrightarrow \quad \boxed{x' + \frac{1}{yLny}x = \frac{1}{y}}$$

Ekuazio lineala, beraz (*zehatzgarria* baita ere. *Ikus:* μ = 1/y). Hurrengo eran askatuko dugu:

$$x = e^{-\int\frac{dy}{yL_n y}}\left[\int Lny\frac{1}{y}dy + C\right] = \frac{1}{Lny}\left[\frac{(Lny)^2}{2} + C\right] = \frac{Lny}{2} + \frac{C}{Lny}$$

edota: $\qquad 2x\cdot L_n y = (Lny)^2 + C$

Hastapen baldintza: $y(1) = e$;

$$2\cdot L_n e = (L_n e)^2 + C \quad \Leftrightarrow \quad 2 = 1 + C \quad \Leftrightarrow \quad C = 1$$

Soluzio partikularra, azkenik: $\qquad 2x\cdot L_n y = (Lny)^2 + 1$

BIGARREN LAUHILABETE

C.1.- Aska ezazu hurrengo hastapen baldintzatako ekuazio diferentziala:

$$x'' + 2x' + 5x = 3e^{-t}\cos 2t; \quad x(0) = x'(0) = 0$$

Laplace-ren $\mathcal{L}$ eragilea aplikatuz, *X(p)* transformatua askatuko da:

$$\left(p^2 + 2p + 5\right)X(p) = \frac{3(p+1)}{(p+1)^2+4} \quad \rightarrow \quad X(p) = \frac{3(p+1)}{\left[(p+1)^2+4\right]^2}$$

Idatz daitekeena: $X(p) = \dfrac{p+1}{(p+1)^2+4} \cdot \dfrac{3}{(p+1)^2+4} = F(p)\cdot G(p)$

Konboluzio teoremarekin baliatuz, hurrengo integrala ebatziz lortuko da *X(p)* –ren alderantzizkoa:

$$x(t) = \mathcal{L}^{-1}[X(p)] = \mathcal{L}^{-1}[F(p)\cdot G(p)] = \int_0^t f(u)\cdot g(t-u)du$$

Aldez aurretik *f(x)* eta *g(x)* *Alderantzizko Taula-ren* bidez kalkulatuko ditugu:

$$f(x) = \mathfrak{L}^{-1}\left[\frac{p+1}{(p+1)^2+4}\right] = e^{-t}cos\,2t \quad ; \quad g(x) = \mathfrak{L}^{-1}\left[\frac{3}{(p+1)^2+4}\right] = \frac{3}{2}\,e^{-t}sin\,2t$$

Azkenik:

$$x(t) = \int_0^t e^{-u}cos\,2u\cdot\frac{3}{2}\,e^{-(t-u)}sin\,2(t-u)\,du = \frac{3}{2}e^{-t}\int_0^t cos\,2u\cdot sin\,2(t-u)\,du =$$

$$x(t) = \frac{3}{2}e^{-t}\int_0^t\left[cos\,2u\,(sin\,2t\cdot cos\,2u - cos\,2t\cdot sin\,2u)\ \right]du =$$

$$x(t) = \frac{3}{2}e^{-t}\left[\,sin\,2t\int_0^t cos^2\,2u\ \,du - cos\,2t\int_0^t\ sin\,2u\ cos\,2u\ du\right] =$$

$$x(t) = \frac{3}{2}e^{-t}\left[\frac{sin\,2t}{2}\left(t + \frac{sin\,2t\cdot cos\,2t}{2}\right) - cos\,2t\ \frac{sin^2\,2t}{2\cdot 2}\right] =$$

$$x(t) = \frac{3}{4}e^{-t}\left[\,t\,sin\,2t + \frac{sin^2\,2t\cdot cos\,2t}{2} - \frac{cos\,2t\cdot sin^2\,2t}{2}\right] = \frac{3}{4}t\,e^{-t}\ sin\,2t$$

C.2.- Kalkulu operazionala erabiliz, aska ezazu hurrengo ekuazio integrala:

$$x(t) = \frac{t^2}{2} - \int_0^t x(u)Sh(t-u)du$$

Laplace-ren $\mathcal{L}$ eragilea aplikatuz, *X(p)* transformatua askatuko da:

$$X(p)=\frac{1}{2}\frac{2}{p^3}-X(p)\cdot\frac{1}{p^2-1}\quad\rightarrow\quad X(p)\left[1+\frac{1}{p^2-1}\right]=\frac{1}{p^3}\quad\rightarrow\quad X(p)=\frac{p^2-1}{p^5}$$

$$X(p)=\frac{1}{p^3}-\frac{1}{p^5}$$

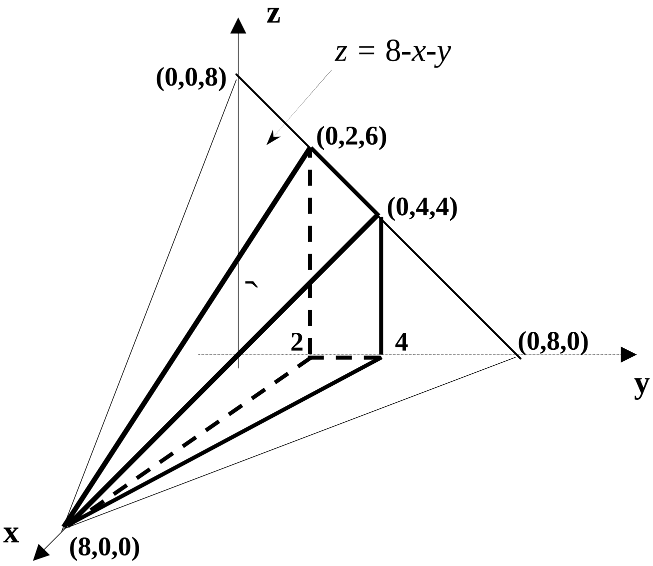
$\mathcal{L}^{-1}$ alderantzizko eragilearekin *x(t)* erdietsiko dugu:

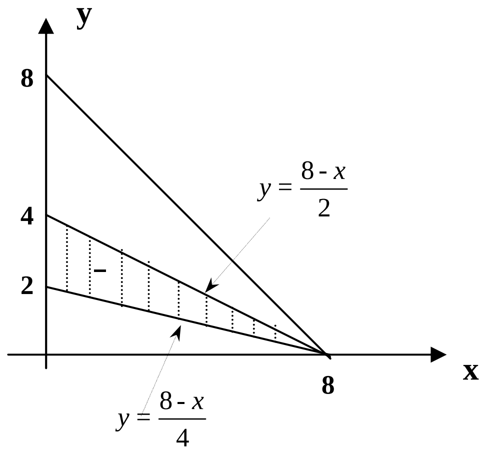
$$x(t)=\mathcal{L}^{-1}\left[X(p)\right]=\mathcal{L}^{-1}\left[\frac{1}{p^3}-\frac{1}{p^5}\right]=\frac{1}{2}t^2-\frac{1}{24}t^4$$

D.- Oktante positiboko [C] gorputz homogenoaren muga-gainazalak hurrengoak dira:

$$[\sigma_1]:\quad x+y+z-8=0;\quad [\sigma_2]:\quad x+4y-8=0\quad [\sigma_3]:\quad x+2y-8=0$$

D.1.- [C] gorputz homogenoaren grabitate-zentru geometrikoaren x_c koordenatua kalkulatu.





XOY plano gaineko [C] gorputzaren proiekzio zilindrikoa.

Grabitate-zentru geometrikoa. x_c koordenatuaren kalkulua (*homogenoa:δ konstantea)*.

$$x_c=\frac{M_{YOZ}}{m_C}=\frac{\delta\iiint_V xdxdydz}{\delta\iiint_V dxdydz}=\frac{\iiint_V xdxdydz}{\iiint_V dxdydz}=\frac{\iiint_V xdxdydz}{V_C}$$

Bolumenaren kalkulu

$$V_C = \iiint_C dx\,dy\,dz = \iint_{D\text{-}XOY} dx\,dy \int_0^{8-x-y} dz = \int_0^8 dx \int_{(8-x)/4}^{(8-x)/2} (8-x-y)\,dy =$$

$$V_C = \int_0^8 \left[\frac{(8-x-y)^2}{-2} \right] \Bigg|_{(8-x)/4}^{(8-x)/2} dx = \frac{-1}{2} \int_0^8 \left[\left(\frac{8-x}{2} \right)^2 - \left(\frac{3(8-x)}{4} \right)^2 \right] dx =$$

$$V_C = \frac{-1}{2} \left[\left(\frac{1}{2} \right)^2 - \left(\frac{3}{4} \right)^2 \right] \int_0^8 (8-x)^2 \ dx = \frac{5}{32} \left[\frac{(8-x)^3}{-3} \right] \Bigg|_0^8 = \frac{5}{32} \cdot \frac{8^3}{3} = \frac{80}{3}$$

YOZ planoarekiko momentuaren kalkulua:

$$M_{YOZ} = \iiint_V xdxdydz = \iint_D xdxdy \int_0^{8-x-y} dz = \int_0^8 xdx \int_{(8-x)/4}^{(8-x)/2} (8-x-y)\,dy =$$

$$M_{YOZ} = \int_0^8 \left[\frac{(8-x-y)^2}{-2} \right] \Bigg|_{(8-x)/4}^{(8-x)/2} xdx = \frac{-1}{2} \int_0^8 \left[\left(\frac{8-x}{2} \right)^2 - \left(\frac{3(8-x)}{4} \right)^2 \right] xdx =$$

$$M_{YOZ} = \frac{-1}{2} \left[\left(\frac{1}{2} \right)^2 - \left(\frac{3}{4} \right)^2 \right] \int_0^8 x(8-x)^2\,dx = \frac{5}{32} \int_0^8 \left(64x - 16x^2 + x^3 \right) dx =$$

$$M_{YOZ} = \frac{5}{32} \left[32x^2 - \frac{16}{3}x^3 + \frac{x^4}{4} \right] \Bigg|_0^8 = 5 \left[64 - \frac{256}{3} + 32 \right] = \frac{160}{3}$$

Azkenik: $$x_c = \frac{M_{YOZ}}{m_{\mathbb{C}}} = \frac{\iiint_V xdxdydz}{V_C} = \frac{160/3}{80/3} = 2\ (u)$$

D.2.- [C] gorputzaren $[\sigma_1]$ muga-gainazal atalaren azalera kalkulatu.

D.2.- $[\sigma_1]$ muga-gainazal atalaren azalera kalkulatu.

OZ-rekiko erregularra da $[\sigma_1]$ gainazal mugatzailearen atala, bere XOY plano gaineko proiekzioa aurreko D.1 –eko integral hirukoitzetako eremua den ¯ triangelu bera da. Integral bikoitza bilakatuz hurrengo eran kalkulatzen da Gainazal Integrala:

$$S_{\sigma_1} = \iint_{\sigma_1} d\sigma = \iint_D \frac{dxdy}{|\cos\gamma|}$$

$$*****[\sigma_1]: \quad x + y + z - 8 = 0 \qquad \rightarrow \qquad \vec{n}_{\sigma_1} = \frac{[1,1,1]}{\sqrt{1^2+1^2+1^2}} \equiv \frac{[1,1,1]}{\sqrt{3}} \quad \rightarrow$$

$$|\cos\gamma| = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad S_{\sigma_1} = \iint_D \frac{dxdy}{|\cos\gamma|} = \iint_D \sqrt{3}\, dxdy = \sqrt{3}\int_0^8 dx \int_{(8-x)/4}^{(8-x)/2} dy =$$

$$S_{\sigma_1} = \sqrt{3}\left(\frac{1}{2} - \frac{1}{4}\right)\int_0^8 (8-x)dx = \frac{\sqrt{3}}{4}\left[\frac{(8-x)^2}{-2}\right]\Bigg|_0^8 = \frac{\sqrt{3}\cdot 8^2}{8} = 8\sqrt{3} \quad (u^2)$$

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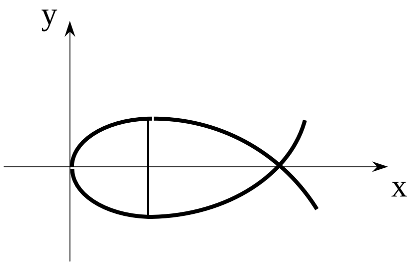
Oinarri Matematikoak I – Azterketa – 2. deialdia
Elektrizitatea eta Elektronika   2004-09-06

LEHEN LAUHILABETE

A.1.- Lehen deribatua eta definizio-eremuaren azterketa analitikoa burutuz, egiaztatu irudikoa dela hurrengo funtzioaren adierazpide grafikoa:

$$9y^2 = x(x-3)^2 \qquad \Rightarrow$$



A.2.- Korapiloaren azalera kalkulatu.

A.3.- Froga ezazu $\mathrm{L} = \int_0^3 \frac{x+1}{\sqrt{x}}\,dx$ integralak irudiko korapiloaren luzeraren balioa adierazten duela. L kalkulatu.

A.4.- Irudiko korapiloak *OX* ardatz inguru biratzean sortzen duen gainazalaren azalera kalkulatu.

B.1.- Biz hurrengo ekuazioaren bidez definitutako $z = z(x, y)$ funtzioa:

$$\mathrm{z}^{-1} + \mathrm{Lny} = \Phi\left(\frac{\mathrm{y} + \mathrm{xLny}}{\mathrm{x}}\right)$$

non (ϕ) funtzio arbitrarioa eta diferentziagarria den. Aurkitu ahalik eta era sinplifikatuenean hurrengo adierazpen diferentzialaren balioa:

$$\mathrm{E} \equiv \mathrm{x}(\mathrm{x}+\mathrm{y})\frac{\partial \mathrm{z}}{\partial \mathrm{x}} + \mathrm{y}^2\frac{\partial \mathrm{z}}{\partial \mathrm{y}} \quad .$$

B.2.- Askatu hurrengo ekuazio diferentziala: $y' = \dfrac{-yLny}{x - Lny} \quad ; \quad y(e) = 1$

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Elektrizitatea eta Elektronika   2004-09-06

BIGARREN LAUHILABETE

B.2.- Askatu hurrengo ekuazio diferentziala: $y' = \dfrac{-yLny}{x - Lny}$; $y(e) = 1$

C.1.- Aska ezazu hurrengo hastapen baldintzatako ekuazio diferentziala:

$$x'' + 2x' + 5x = 3e^{-t}\cos 2t; \quad x(0) = x'(0) = 0$$

C.2.- Kalkulu operazionala erabiliz, aska ezazu hurrengo ekuazio integrala:

$$x(t) = \frac{t^2}{2} - \int_0^t x(u)Sh(t-u)du$$

D.- Oktante positiboko [˜] gorputz homogenoaren muga-gainazalak hurrengoak dira:

$$[\sigma_1]: \quad x+y+z-8=0; \quad [\sigma_2]: \quad x+4y-8=0 \quad [\sigma_3]: \quad x+2y-8=0$$

D.1.- [˜] gorputzaren grabitate-zentru geometrikoaren x_c koordenatua kalkulatu.

D.2.- [˜] gorputzaren $[\sigma_1]$ muga-gainazal atalaren azalera kalkulatu.

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JARRAIBIDEAK

- Mota guztietako bibliografia erabilgarria da.
- Azterketa **partzial bakoitzaren** iraupena **:** **Ordubete eta hiru laurden.** *(1o. 45m.)*
- Azterketa **osoaren** iraupena **:** **Hiru ordubete eta erdi.** *(3o. 30m.)*

Bananduta aurkeztuko dira ariketak, ondorengo ordenean
A orria: [A.1 ; A.2 ; A.3 ; A.4 *ariketak*], azterketa hasi eta **ORDUBETERA.** *(60m.)*
B orria: [B.1 ; B.2 *ariketak*], azt. hasi e. **ORDUBETE eta 3 LAURDENERA.** *(45m)*
C orria: [C.1 ; C.2 *ariketak*], a. h. eta 2 **ORDUBETE eta 3 LAURDENERA.** *(60m.)*
D orria: [D.1 ; D.2 *ariketak*], a. h. eta 3 **ORDUBETE eta ERDIRA.** *(45m.)*

- Azterketaren guztirako puntuaketa: **40 puntu (orriko, 10 puntu).**
- Kalifikazioen publikazioa: **Irailaren 10ean, eguerdiko 13etan.** *(***5***n. solairuan).*
- Berrikuspena: **Irailaren 13an, eguerdiko 12etan**. *(***5***n. solairuko* **51** *ikasgelan).*

Oinarri Matematikoak I – 2. Partziala - Azterketa FINALA
1. ATALA Elektrizitate eta Elektronika 2005-05-30

B.2.- Laplace-ren eragile lineala aplikatu ondoko ekuazio diferentzialetako hastapen-baldintzatako sistema askatzeko:

$$\begin{cases} x'(t) - 2x(t) + y(t) = 1 \\ y'(t) + 4x(t) + y(t) = e^t \\ x(0) = y(0) = 1 \end{cases}$$

10 puntu

Transformatu zuzena:

$$\begin{cases} p\cdot \mathrm{X}(p) - 1 - 2\cdot \mathrm{X}(p) + \mathrm{Y}(p) = \dfrac{1}{p} \\ p\cdot \mathrm{Y}(p) - 1 + 4\cdot \mathrm{X}(p) + \mathrm{Y}(p) = \dfrac{1}{p-1} \end{cases} ; \begin{cases} (p-2)\cdot \mathrm{X}(p) + \mathrm{Y}(p) = \dfrac{1}{p} + 1 = \dfrac{p+1}{p} \\ 4\cdot \mathrm{X}(p) + (p+1)\cdot \mathrm{Y}(p) = \dfrac{1}{p-1} + 1 = \dfrac{p}{p-1} \end{cases}$$

$$\mathrm{X}(p) = \frac{\begin{vmatrix} \dfrac{p+1}{p} & 1 \\ \dfrac{p}{p-1} & p+1 \end{vmatrix}}{\begin{vmatrix} p-2 & 1 \\ 4 & p+1 \end{vmatrix}} = \frac{\dfrac{(p+1)^2}{p} - \dfrac{p}{p-1}}{p^2 - p - 6} = \frac{(p+1)^2(p-1) - p^2}{p(p-1)(p-3)(p+2)} =$$

$$\mathrm{X}(p) = \frac{p^3 - p - 1}{p(p-1)(p-3)(p+2)} = \frac{A}{p} + \frac{B}{p-1} + \frac{D}{p-3} + \frac{E}{p+2}$$

$$p^3 - p - 1 = A(p-1)(p-3)(p+2) + Bp(p-3)(p+2) + Dp(p-1)(p+2) + Ep(p-1)(p-3)$$

$$\begin{cases} p = 0 : & -1 = 6A \\ p = 1 : & -1 = -6B \\ p = 3 : & 23 = 30D \\ p = -2 : & -7 = -30E \end{cases} \Leftrightarrow \begin{cases} A = -1/6 \\ B = \ 1/6 \\ D = 23/30 \\ E = \ 7/30 \end{cases}$$

$$x(t) = \mathfrak{L}^{-1}\left[\frac{-1/6}{p} + \frac{1/6}{p-1} + \frac{23/30}{p-3} + \frac{7/30}{p+2}\right] = \frac{-1}{6} + \frac{1}{6}e^t + \frac{23}{30}e^{3t} + \frac{7}{30}e^{-2t} = \frac{1}{30}\left(-5 + 5e^t + 23e^{3t} + 7e^{-2t}\right)$$

Lehen ekuaziotik:

$$y(t) = 1 - x'(t) + 2x(t) = 1 - \frac{1}{30}\left(5e^t + 69e^{3t} - 14e^{-2t}\right) + \frac{2}{30}\left(-5 + 5e^t + 23e^{3t} + 7e^{-2t}\right)$$

azkenik: $$y(t) = \frac{20}{30} + \frac{5}{30}e^t - \frac{23}{30}e^{3t} + \frac{28}{30}e^{-2t} = \frac{1}{30}\left(20 + 5e^t - 23e^{3t} + 28e^{-2t}\right)$$

$$x(t) = = \frac{1}{30}\left(-5 + 5e^t + 23e^{3t} + 7e^{-2t}\right)$$

B.3.- Integratu $y' = \dfrac{y^4 + 4(x+1)}{2y^3}$ ekuazio diferentziala. **12 puntu**

Zehatzgarria da, zeren: $\dfrac{dy}{dx} = \dfrac{y^4 + 4(x+1)}{2y^3} \Leftrightarrow \left[y^4 + 4(x+1)\right]dx - 2y^3 dy = 0$

$$\frac{\partial P}{\partial y} = \frac{\partial\left[y^4 + 4(x+1)\right]}{\partial y} = 4y^3 \quad ; \quad \frac{\partial Q}{\partial x} = \frac{\partial\left[-2y^3\right]}{\partial x} = 0$$

Fakore integrantea: $\dfrac{P'_y - Q'_x}{Q} = \dfrac{4y^3 - 0}{-2y^3} = -2 \quad \Rightarrow \quad \boxed{\mu(x) = e^{-\int 2dx} = e^{-2x}}$

Fakore integrantea erabiliz: $\left[y^4 + 4(x+1)\right]e^{-2x}dx - 2y^3 e^{-2x}dy = 0$

Funtzio Potentziala:

$$\mathrm{U}(x,y)\text{-}\mathrm{U}(0,0) = \int_{\widehat{AB}} d\mathbf{U} = \int_{\widehat{AB}} \left[y^4 + 4(x+1)\right]e^{-2x}dx - 2y^3e^{-2x}dy = \int_{(0,0)}^{(x,0)} \left[y^4 + 4(x+1)\right]e^{-2x}dx - 2e^{-2x}\int_{(x,0)}^{(x,y)} y^3 dy =$$

$$I_1 = \int_{(0,0)}^{(x,0)} \left[y^4 + 4(x+1)\right]e^{-2x}dx = \left[y^4 \frac{e^{-2x}}{-2}\right]\Bigg|_{(0,0)}^{(x,0)} + 4\int_{(0,0)}^{(x,0)} (x+1)e^{-2x}dx = \left[0-0\right] - e^{-2x}\left(2x+3\right) = -e^{-2x}\left(2x+3\right)$$

$$I_2 = -2\int_{(x,0)}^{(x,y)} y^3 e^{-2x}dy = -2\left(e^{-2x}\frac{y^4}{4}\right)\Bigg|_{(x,0)}^{(x,y)} = \frac{-1}{2}\left(e^{-2x}y^4 - 0\right) = \frac{-1}{2}e^{-2x}y^4$$

Azkenik: $\mathrm{U}(x,y)\text{-}\mathrm{U}(0,0) = -e^{-2x}\left(2x+3\right) - \dfrac{1}{2}e^{-2x}y^4 = -\dfrac{e^{-2x}}{2}\left(4x+6+y^4\right) \Leftrightarrow \boxed{e^{-2x}\left(4x+6+y^4\right) = \mathrm{C}}$

BERNOULLI:

$$y' - \frac{y}{2} = 2(x+1)y^{-3} \qquad y^4 = 4z \qquad \begin{array}{l} y = 4zy^{-3} \\ y' = z'y^{-3} \end{array}$$

$$z'y^{-3} - \frac{4zy^{-3}}{2} = 2(x+1)y^{-3} \Leftrightarrow z' - 2z = 2(x+1) \quad ; \quad u(x) = e^{\int 2dx} = e^{2x}$$

$$z(x) = u(x)\cdot v(x) \,/ \quad v(x) = 2\int e^{-2x}(x+1) \quad ; \quad \begin{vmatrix} u = (x+1) & du = dx \\ dv = 2e^{-2x}dx & v = -e^{-2x} \end{vmatrix}$$

,

$$v(x) = \left[-(x+1)e^{-2x} - \frac{1}{2}e^{-2x}\right] + C = -e^{-2x}\left(x + \frac{3}{2}\right) + C$$

$$\textit{Beraz :} \qquad z(x) = u(x)\cdot v(x) = e^{2x}\left[-e^{-2x}\left(x+\frac{3}{2}\right) + C\right] = Ce^{2x} - \frac{1}{2}\left(2x+3\right)$$

$$\textit{Aldagai-berreskuraketa:} \quad y^4 = 4z \quad \Leftrightarrow \quad y^4 = 4\left[Ce^{2x} - \frac{1}{2}\left(2x+3\right)\right] = 4Ce^{2x} - \left(4x+6\right)$$

$$\textit{Azkenik :} \quad \boxed{y^4 = Ce^{2x} - \left(4x+6\right)}$$

Oinarri Matematikoak I –- Azterketa FINALA
1. ATALA Elektrizitate eta Elektronika 2005-05-30

A.1.- Froga ezazu $x^2+y^2-2\sqrt{3}y-9=0$ **zirkunferentzia** ondoko baldintza betetzen duten plano konplexuko puntuetako toki geometrikoa dela:

$$\arg\left(\frac{z-3}{z+3}\right)=\frac{\pi}{3}$$

6 puntu

Soluzioa

$$\frac{z-3}{z+3}=\frac{(x-3)+yi}{(x+3)+yi}=\frac{[(x-3)+yi][(x+3)-yi]}{[(x+3)+yi][(x+3)-yi]}=\frac{\left[(x^2-3^2)+y^2\right]+iy\left[(x+3)-(x-3)\right]}{(x+3)^2+y^2}=$$

$$\frac{z-3}{z+3}=\frac{\left[(x^2-9)+y^2\right]+i[6y]}{(x+3)^2+y^2}=\frac{(x^2-9)+y^2}{(x+3)^2+y^2}+i\frac{6y}{(x+3)^2+y^2}=a+bi$$

$$\arg\left(\frac{z-3}{z+3}\right)=\arg(a+bi)=\frac{\pi}{3}\quad\Rightarrow\quad tg\frac{\pi}{3}=\frac{b}{a}=\frac{6y}{(x^2-9)+y^2}=\sqrt{3}$$

Azkenik, toki geometrikoaren ekuazioa:

$$(x^2-9)+y^2=\frac{6y}{\sqrt{3}}=2\sqrt{3}y\quad\Leftrightarrow\quad\boxed{x^2+y^2-2\sqrt{3}y-9=0}\quad f..n.b$$

A.2.- $y=ax^3+bx^2+cx+d$ **,** $A(0,3)$ eta $B(2,5)$ puntuetatik pasatzen den lerro baten ekuazioa da. $C(1,10)$ puntuko ukitzailea horizontala izanik, lortu (y)-ren ekuazioa.
(y)-ren extremo erlatibo eta inflexio puntuak aztertu, eta dagokionean kalkulatu.

7 puntu

Soluzioa:

$A(0,3)$: $3=a\cdot 0^3+b\cdot 0^2+c\cdot 0+d \quad\Leftrightarrow\quad 3=d$

$B(2,5)$: $5=a\cdot 2^3+b\cdot 2^2+c\cdot 2+d \quad\Leftrightarrow\quad 8a+4b+2c+d=5$

$C(1,10)$: $10=a\cdot 1^2+b\cdot 1+c\cdot 1+d \quad\Leftrightarrow\quad a+b+c+d=10$

$y'(1)=0$: $0=3a\cdot 1^2+2b\cdot 1+c \quad\Leftrightarrow\quad 3a+2b+c=0$

a **=1**
b **= -9**
c **=15**
d **=3**

Lerroaren ekuazioa: $\boldsymbol{y = x^3 - 9x^2 + 15x + 3}$

Puntu singularrak: $y'=3x^2-18x+15=0 \quad\Rightarrow\quad x=1 \;\wedge\; x=5$

Sailkapena: $y''=6x-18\Rightarrow\begin{cases}x=1; & y''(1)=-12<0 \;\Rightarrow\; \text{Maximo erlatibo}:(1,10)\\ x=5; & y''(5)=\;\;12>0 \;\Rightarrow\; \text{Minimo erlatibo}:(5,-22)\end{cases}$

Inflexio-puntu: $y''=6x-18=0 \quad\Rightarrow\quad x=3 \quad\Rightarrow\quad D(3,-6)$

A.3.- Ebatz ezazu $I=\int \cos^3 x\sqrt{1+\sin x}\,dx$ integrala. **6 puntu**

Soluzioa: $I=\int \cos^3 x\sqrt{1+\sin x}\,dx=\int \cos x\cdot\cos^2 x\sqrt{1+\sin x}\,dx$; $\left|\begin{matrix}1+\sin x=t^2\\ \cos x\,dx=2t\,dt\end{matrix}\right|$;

$$I=\int\left[1-(t^2-1)^2\right]\cdot t\cdot 2t\,dt=2\int(2t^4-t^6)\,dt=2\left[2\frac{t^5}{5}-\frac{t^7}{7}\right]+C=$$

$$I=\left[4\frac{t^5}{5}-2\frac{t^7}{7}+C\right]=\boxed{\frac{4}{5}(1+\sin x)^{5/2}-\frac{2}{7}(1+\sin x)^{7/2}+C}$$

B.1.- $z^3+xy^2=x$ ekuazioaren bidez definitutako $z(x,y)$ funtzio inplizituaren hessiarra H izanik, $\underline{z^4\cdot H}$ kalkulatu. **6 puntu**

$$\boxed{z^3=x(1-y^2)}\qquad H(z)=\begin{vmatrix}\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x\partial y}\\ \frac{\partial^2 z}{\partial y\partial x} & \frac{\partial^2 z}{\partial y^2}\end{vmatrix}=\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2\qquad \boxed{z_x=\frac{1-y^2}{3z^2}=\frac{z}{3x}}$$

1) $$z^3+xy^2=x\xrightarrow[\partial x]{}3z^2\cdot z_x+y^2=1\xrightarrow[\partial x]{}6z\cdot z_x\cdot z_x+3z^2\cdot z_{xx}=0$$

hots: $$2(z_x)^2+z\cdot z_{xx}=0\ \Rightarrow\ \boxed{z_{xx}}=\frac{-2(z_x)^2}{z}=\frac{-2(z)^2}{z\cdot(3x)^2}=\boxed{\frac{-2z}{9x^2}}$$

2) $$z^3+xy^2=x\xrightarrow[\partial y]{}3z^2\cdot z_y+2xy=0\xrightarrow[\partial y]{}6z\cdot z_y\cdot z_y+3z^2\cdot z_{yy}+2x=0;$$

$$z_{yy}=\frac{-2(z_y)^2}{z}-\frac{2x}{3z^2}$$

$$\boxed{z_{yy}}=\frac{-2(-2xy/3z^2)^2}{z}-\frac{2x}{3z^2}=\frac{-8x^2y^2}{9z^5}-\frac{2x}{3z^2}=\frac{-8x^2(x-z^3)}{9z^5x}-\frac{2x}{3z^2}=\boxed{\frac{-8x^2}{9z^5}+\frac{2x}{9z^2}}$$

3) $$z^3+xy^2=x\xrightarrow[\partial x]{}3z^2\cdot z_x+y^2=1\xrightarrow[\partial y]{}6z\cdot z_y\cdot z_x+3z^2\cdot z_{xy}+2y=0$$

hots: $$\boxed{z_{xy}}=\frac{-2z_y\cdot z_x}{z}-\frac{2y}{3z^2}=\frac{-2}{z}\cdot\frac{z}{3x}\cdot\frac{(-2xy)}{3z^2}-\frac{2y}{3z^2}=\frac{4y}{9z^2}-\frac{2y}{3z^2}=\boxed{\frac{-2y}{9z^2}}$$

$$H(z)=\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=\frac{-2z}{9x^2}\left(\frac{-8x^2}{9z^5}+\frac{2x}{9z^2}\right)-\left(\frac{-2y}{9z^2}\right)^2=\frac{16}{81z^4}-\frac{4}{81xz}-\frac{4y^2}{81z^4}$$

$$H(z)=\frac{12+4(1-y^2)}{81z^4}-\frac{4}{81xz}=\frac{12}{81z^4}+\frac{4z^3}{81z^4x}-\frac{4}{81xz}=\frac{4}{27z^4}$$

$$\boxed{z^4\cdot H}=z^4\cdot\frac{4}{27z^4}=\boxed{\frac{4}{27}}$$

B.2.- Laplace-ren eragile lineala aplikatu ondoko ekuazio diferentzialetako hastapen-baldintzatako sistema askatzeko:

$$\begin{cases} x'(t) - 2x(t) + y(t) = 1 \\ y'(t) + 4x(t) + y(t) = e^t \\ x(0) = y(0) = 1 \end{cases}$$

7 puntu

Transformatu zuzena:

$$\begin{cases} p \cdot \mathrm{X}(p) - 1 - 2 \cdot \mathrm{X}(p) + \mathrm{Y}(p) = \dfrac{1}{p} \\ p \cdot \mathrm{Y}(p) - 1 + 4 \cdot \mathrm{X}(p) + \mathrm{Y}(p) = \dfrac{1}{p-1} \end{cases} ; \begin{cases} (p-2) \cdot \mathrm{X}(p) + \mathrm{Y}(p) = \dfrac{1}{p} + 1 = \dfrac{p+1}{p} \\ 4 \cdot \mathrm{X}(p) + (p+1) \cdot \mathrm{Y}(p) = \dfrac{1}{p-1} + 1 = \dfrac{p}{p-1} \end{cases}$$

$$\mathrm{X}(p) = \frac{\begin{vmatrix} \dfrac{p+1}{p} & 1 \\ \dfrac{p}{p-1} & p+1 \end{vmatrix}}{\begin{vmatrix} p-2 & 1 \\ 4 & p+1 \end{vmatrix}} = \frac{\dfrac{(p+1)^2}{p} - \dfrac{p}{p-1}}{p^2 - p - 6} = \frac{(p+1)^2(p-1) - p^2}{p(p-1)(p-3)(p+2)} =$$

$$\mathrm{X}(p) = \frac{p^3 - p - 1}{p(p-1)(p-3)(p+2)} = \frac{A}{p} + \frac{B}{p-1} + \frac{D}{p-3} + \frac{E}{p+2}$$

$$p^3 - p - 1 = A(p-1)(p-3)(p+2) + Bp(p-3)(p+2) + Dp(p-1)(p+2) + Ep(p-1)(p-3)$$

$$\begin{cases} p = 0 \ : \ \ -1 = 6A \\ p = 1 \ : \ \ -1 = -6B \\ p = 3 \ : \ \ 23 = 30D \\ p = -2 : \ -7 = -30E \end{cases} \Leftrightarrow \begin{cases} A = -1/6 \\ B = \ \ 1/6 \\ D = 23/30 \\ E = \ \ 7/30 \end{cases}$$

$$x(t) = \mathfrak{L}^{-1}\left[\frac{-1/6}{p} + \frac{1/6}{p-1} + \frac{23/30}{p-3} + \frac{7/30}{p+2}\right] = \frac{-1}{6} + \frac{1}{6}e^t + \frac{23}{30}e^{3t} + \frac{7}{30}e^{-2t} = \frac{1}{30}\left(-5 + 5e^t + 23e^{3t} + 7e^{-2t}\right)$$

Lehen ekuaziotik:

$$y(t) = 1 - x'(t) + 2x(t) = 1 - \frac{1}{30}\left(5e^t + 69e^{3t} - 14e^{-2t}\right) + \frac{2}{30}\left(-5 + 5e^t + 23e^{3t} + 7e^{-2t}\right)$$

azkenik: $$y(t) = \frac{20}{30} + \frac{5}{30}e^t - \frac{23}{30}e^{3t} + \frac{28}{30}e^{-2t} = \frac{1}{30}\left(20 + 5e^t - 23e^{3t} + 28e^{-2t}\right)$$

$$x(t) = = \frac{1}{30}\left(-5 + 5e^t + 23e^{3t} + 7e^{-2t}\right)$$

B.3.- Integratu $y' = \dfrac{y^4 + 4(x+1)}{2y^3}$ ekuazio diferentziala. **6 puntu**

Zehatzgarria da, zeren: $\dfrac{dy}{dx} = \dfrac{y^4 + 4(x+1)}{2y^3} \Leftrightarrow \left[y^4 + 4(x+1)\right]dx - 2y^3 dy = 0$

$$\frac{\partial P}{\partial y} = \frac{\partial\left[y^4 + 4(x+1)\right]}{\partial y} = 4y^3 \quad ; \quad \frac{\partial Q}{\partial x} = \frac{\partial\left[-2y^3\right]}{\partial x} = 0$$

Fakore integrantea: $\dfrac{P'_y - Q'_x}{Q} = \dfrac{4y^3 - 0}{-2y^3} = -2 \quad \Rightarrow \quad \boxed{\mu(x) = e^{-\int 2dx} = e^{-2x}}$

Fakore integrantea erabiliz: $\left[y^4 + 4(x+1)\right]e^{-2x}dx - 2y^3 e^{-2x}dy = 0$

Funtzio Potentziala:

$$\mathrm{U}(x,y)-\mathrm{U}(0,0) = \int_{\widehat{AB}} d\mathbf{U} = \int_{\widehat{AB}} \left[y^4 + 4(x+1)\right]e^{-2x}dx - 2y^3e^{-2x}dy = \int_{(0,0)}^{(x,0)}\left[y^4 + 4(x+1)\right]e^{-2x}dx - 2e^{-2x}\int_{(x,0)}^{(x,y)} y^3 dy =$$

$$I_1 = \int_{(0,0)}^{(x,0)}\left[y^4 + 4(x+1)\right]e^{-2x}dx = \left[y^4\frac{e^{-2x}}{-2}\right]\Bigg|_{(0,0)}^{(x,0)} + 4\int_{(0,0)}^{(x,0)}(x+1)e^{-2x}dx = \left[0-0\right] - e^{-2x}\left(2x+3\right) = -e^{-2x}\left(2x+3\right)$$

$$I_2 = -2\int_{(x,0)}^{(x,y)} y^3e^{-2x}dy = -2\left(e^{-2x}\frac{y^4}{4}\right)\Bigg|_{(x,0)}^{(x,y)} = \frac{-1}{2}\left(e^{-2x}y^4 - 0\right) = \frac{-1}{2}e^{-2x}y^4$$

Azkenik: $\mathrm{U}(x,y)-\mathrm{U}(0,0) = -e^{-2x}\left(2x+3\right) - \dfrac{1}{2}e^{-2x}y^4 = -\dfrac{e^{-2x}}{2}\left(4x+6+y^4\right) \Leftrightarrow \boxed{e^{-2x}\left(4x+6+y^4\right) = \mathrm{C}}$

BERNOULLI:

$$y' - \frac{y}{2} = 2(x+1)y^{-3} \qquad y^4 = 4z \qquad \begin{array}{l} y = 4zy^{-3} \\ y' = z'y^{-3} \end{array}$$

$$z'y^{-3} - \frac{4zy^{-3}}{2} = 2(x+1)y^{-3} \Leftrightarrow z' - 2z = 2(x+1) \quad ; \quad u(x) = e^{\int 2dx} = e^{2x}$$

$$z(x) = u(x)\cdot v(x) \,/\quad v(x) = 2\int e^{-2x}(x+1) \quad ; \quad \begin{vmatrix} u = (x+1) & du = dx \\ dv = 2e^{-2x}dx & v = -e^{-2x} \end{vmatrix}$$

,

$$v(x) = \left[-(x+1)e^{-2x} - \frac{1}{2}e^{-2x}\right] + C = -e^{-2x}\left(x + \frac{3}{2}\right) + C$$

Beraz : $z(x) = u(x)\cdot v(x) = e^{2x}\left[-e^{-2x}\left(x+\dfrac{3}{2}\right) + C\right] = Ce^{2x} - \dfrac{1}{2}(2x+3)$

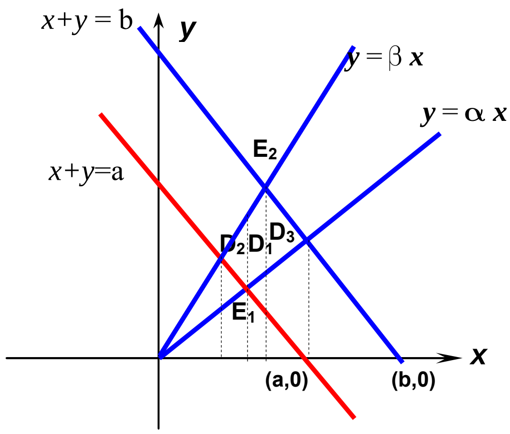
Aldagai-berreskuraketa: $y^4 = 4z \Leftrightarrow y^4 = 4\left[Ce^{2x} - \dfrac{1}{2}(2x+3)\right] = 4Ce^{2x} - (4x+6)$

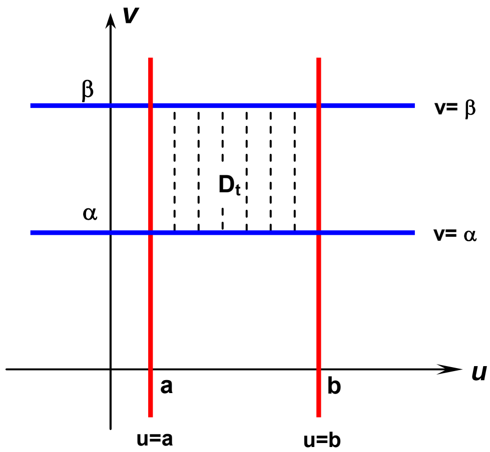
Azkenik : $\boxed{y^4 = Ce^{2x} - (4x+6)}$

C.1.- Hurrengo lau lerroek mugatutako eremuaren azaleraren balioa

$$x+y=a;\quad x+y=b;\quad y=\alpha x;\quad y=\beta x,\qquad (b>a>0\ \ eta\ \beta>\alpha>0)$$

hau dela frogatu: $$A=\frac{(\beta-\alpha)\left(b^2-a^2\right)}{2(\alpha+1)(\beta+1)}$$ **11 puntu**





Mugak:

$$\begin{cases}u=x+y\\ v=y/x\end{cases};\qquad non:\begin{cases}u=a\\ u=b\end{cases}\quad eta\quad\begin{cases}v=\alpha\\ v=\beta\end{cases}\qquad\begin{cases}x=u/(v+1)\\ y=uv/(v+1)\end{cases}$$

$$|J(x,y)|=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left\|\begin{matrix}1/(v+1) & -u/(v+1)^2\\ v/(v+1) & u/(v+1)^2\end{matrix}\right\|=\frac{u}{(v+1)^3}\left\|\begin{matrix}1 & -1\\ v & 1\end{matrix}\right\|=\frac{u}{(v+1)^2}$$

$$A_D=\iint_D dxdy=\iint_{D_t}\frac{u}{(v+1)^2}\cdot dudv=\int_a^b udu\int_\alpha^\beta (v+1)^{-2}dv=\left[\frac{u^2}{2}\right]\Bigg|_a^b\cdot\left[\frac{-1}{v+1}\right]\Bigg|_\alpha^\beta=$$

Azalera:

$$A_D=\frac{1}{2}\left(b^2-a^2\right)\left[\frac{-1}{\beta+1}+\frac{1}{\alpha+1}\right]=\left[\frac{\left(b^2-a^2\right)(\beta-\alpha)}{2(\alpha+1)(\beta+1)}\right]\quad u^2$$

C.2.- Biz ondorengo gainazalek mugatutako ***C*** gorputz homogenoa:

$$[\sigma_1]:\ x^2+y^2-(z-4)^2=0\ \ (z\le 4);\ [\sigma_2]:\ x^2+y^2-z-2=0\ \ (z\ge 0)\ \ [\sigma_3]:\ z=0$$

C gorputzaren grabitate zentru geometrikoa $G\left(0,0,\frac{20}{13}\right)$ puntua dela, egiazta itzazu. **11puntu**

σ_1 ***: forma inplizitua - (0,0,4) erpineko Gainazal-konikoaren azpiko atala***

$$x^2+y^2-(z-4)^2=0\quad\Rightarrow\quad z=4-\sqrt{x^2+y^2}=4-\rho$$

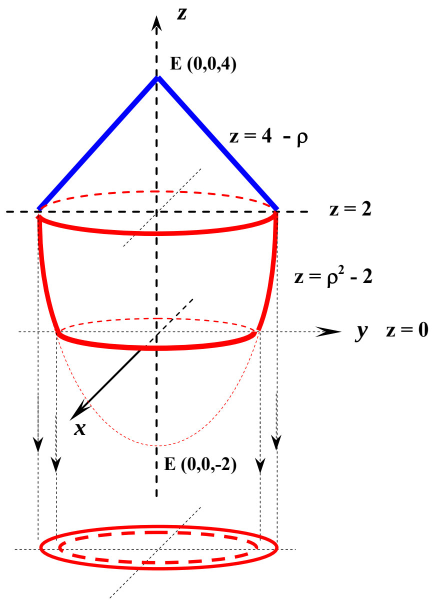
σ_2 ***: forma inplizitua - (0,0,-2) erpineko Paraboloide atala***

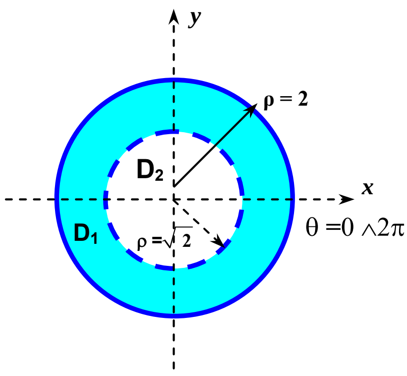
$$x^2+y^2-z-2=0\quad\Rightarrow\quad z=-2+(x^2+y^2)=\rho^2-2$$

Elkarguneak (zirkunferentziak dira, OZ biraketa-ardatz amankomuna duteneko gainazalak baitira) :

$\sigma_1\wedge\sigma_2$: $z=4-\rho=\rho^2-2\Rightarrow\rho^2+\rho-6=0\Rightarrow\ \boldsymbol{\rho=2}\wedge\rho=-3$

$\sigma_2\wedge\sigma_3$: $z=\rho^2-2=0\Rightarrow\quad\rho=\sqrt{\mathbf{2}}$





C gorputzaren Proiekzio zilindrikoa

3. ***Bolumenaren kalkulu:***

$$B_C = \iiint_C dx\,dy\,dz = \iiint_C \rho\,d\rho\,d\theta\,dz = \iiint_{C_1} \rho\,d\rho\,d\theta\,dz + \iiint_{C_2} \rho\,d\rho\,d\theta\,dz =$$

$$B_C = \iint_{D_1} \rho\,d\theta\,d\rho \int_{\rho^2-2}^{4-\rho} dz + \iint_{D_2} \rho\,d\theta\,d\rho \int_0^{4-\rho} dz = \int_0^{2\pi} d\rho \int_{\sqrt{2}}^{2} (6-\rho-\rho^2)\,\rho\,d\rho + \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (4-\rho)\,\rho\,d\rho$$

$$B_C = 2\pi\int_{\sqrt{2}}^{2} (6\rho-\rho^2-\rho^3)\,d\rho + 2\pi\int_0^{\sqrt{2}} (4\rho-\rho^2)\,d\rho = 2\pi\left(3\rho^2-\frac{\rho^3}{3}-\frac{\rho^4}{4}\right)\Bigg|_{\sqrt{2}}^{2} + 2\pi\left(2\rho^2-\frac{\rho^3}{3}\right)\Bigg|_0^{\sqrt{2}} =$$

$$B_C = 2\pi\left[\left(12-\frac{8}{3}-4\right)-\left(6-\frac{2\sqrt{2}}{3}-1\right)+\left(4-\frac{2\sqrt{2}}{3}\right)-(0)\right] = 2\pi\left(7-\frac{8}{3}\right) = \boxed{\frac{26}{3}\pi\ u^3}$$

2. ***Masa-zentruaren kalkulu:***

$$\mathbf{M_{XOY}} = \iiint_C \mathbf{z\,dx\,dy\,dz} = \iiint_C \boldsymbol{\rho\,z\,d\rho\,d\theta\,dz} = = \iiint_{C_1} \rho\,d\rho\,d\theta\,zdz + \iiint_{C_2} \rho\,d\rho\,d\theta\,zdz =$$

$$\mathbf{M_{XOY}} = \iint_{D_1} \rho\,d\theta\,d\rho \int_{\rho^2-2}^{4-\rho} z\,dz + \iint_{D_2} \rho\,d\theta\,d\rho \int_0^{4-\rho} z\,dz = \int_0^{2\pi} d\theta \int_{\sqrt{2}}^{2} \left[z^2/2\right]\Big|_{\rho^2-2}^{4-\rho} \rho\,d\rho + \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \left[z^2/2\right]\Big|_0^{4-\rho} \rho\,d\rho =$$

$$\mathbf{M_{XOY}} = \pi\int_{\sqrt{2}}^{2} \left[(4-\rho)^2-(\rho^2-2)^2\right]\rho\,d\rho + \pi\int_0^{\sqrt{2}} \left[(4-\rho)^2-0^2\right]\rho\,d\rho = \pi\int_0^{2} (4-\rho)^2\rho\,d\rho - \pi\int_{\sqrt{2}}^{2} (\rho^2-2)^2\rho\,d\rho =$$

$$\mathbf{M_{XOY}} = \pi\int_0^{2} \left[16\rho-8\rho^2+\rho^3\right]d\rho - \pi\left[\frac{(\rho^2-2)^3}{3\cdot 2}\right]\Bigg|_{\sqrt{2}}^{2} = \pi\left[\left(16\frac{\rho^2}{2}-8\frac{\rho^3}{3}+\frac{\rho^4}{4}\right)\Bigg|_0^{2}\right] - \frac{\pi}{6}\left[8-0\right] =$$

$$\mathbf{M_{XOY}} = \pi\left[\left(32-\frac{64}{3}+4\right)-(0)-\frac{4}{3}\right] = \boxed{\frac{40}{3}\pi}$$

Masa-zentru (OZ: *simetria-ardatz*): $z_G = \frac{M_{XOY}}{m} = \frac{40\pi/3}{26\pi/3} = \boxed{\frac{20}{13}\ u}$; $G = \boxed{\left(\mathbf{0,0,\frac{20}{13}}\right)}$

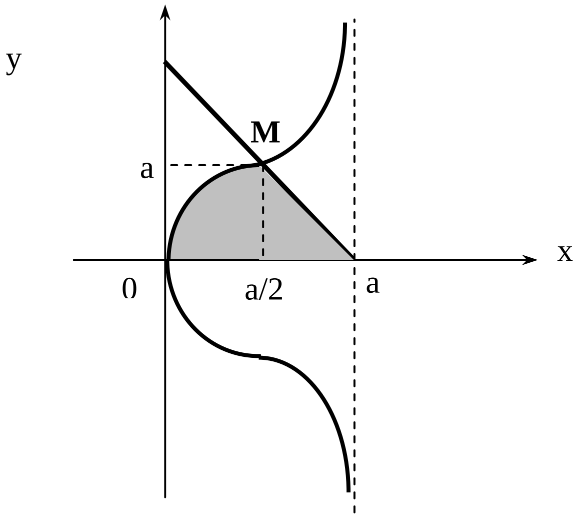
TERCERA PARTE

C.1.-Demostrar que el área D que limitan en el primer cuadrante las líneas,

$$y^2 = \frac{a^2 x}{a-x}; \quad 2x + y = 2a; \quad y = 0, \quad \text{es} \quad A = \frac{a^2(\pi - 1)}{4}$$

Probar que si D gira alrededor de OX genera un cuerpo de volumen: $V = \dfrac{3\ln 2 - 1}{3}\pi a^3$

Indicación: Las líneas se cortan en el punto $M\left(a/2, a\right)$ [11 PUNTOS]



1.- Cálculo del área del dominio D

$$A_D = \iint_D dxdy = \int_0^a dy \int_{ay^2/(a^2+y^2)}^{(2a-y)/2} dx = \int_0^a \left[\frac{(2a-y)}{2} - \frac{ay^2}{a^2+y^2}\right] dy = \frac{1}{2}\left[\frac{(2a-y)^2}{-2}\right]\Bigg|_0^a +$$

$$+ \int_0^a \left[-a + \frac{a^3}{a^2+y^2}\right] dy = \frac{-1}{4}\left[a^2 - 4a^2\right] + \left[-ay + \frac{a^3}{a} arctg\frac{x}{a}\right]\Bigg|_0^a =$$

$$A_D = \frac{3a^2}{4} - a^2 + a^2 \cdot arctg 1 = \frac{-a^2}{4} + \frac{a^2\pi}{4} = \frac{a^2}{4}(\pi - 1)$$

2.- Cálculo del Volumen del cuerpo de revolución alrededor de OX

$$V_C = \pi\int_0^{a/2} y^2 dx + \pi\int_{a/2}^a y^2 dx = \pi\int_0^{a/2} \frac{a^2 x}{a-x} dx + \pi\int_{a/2}^a \left[2(a-x)\right]^2 dx =$$

$$V_C = \pi\int_0^{a/2}\left[-a^2 + \frac{a^3}{a-x}\right]dx + 4\pi\int_{a/2}^a (a-x)^2 dx = \pi\left[-a^2 x - a^3 L_n|a-x|\right]\Bigg|_0^{a/2} +$$

$$+ 4\pi\left[\frac{(a-x)^3}{-1}\right]\Bigg|_{a/2}^a = \pi\left[\frac{-a^3}{2} - a^3 L_n\frac{a}{2} + a^3 L_n a\right] + 4\pi\left[0 + \frac{a^3}{8}\right] = \pi a^3 L_n 2$$

C.2.- Un cuerpo $[V]$ situado en el semiespacio $y \geq 0$ está limitado por las superficies:

$$[\sigma_c]: \quad x^2+z^2=9\,; \quad [\sigma_p]: \quad x^2+9\left(z^2-y\right)=0 \quad [\sigma_3]: \quad y=0.$$

---

1. Mostrar que su volumen es $V=\dfrac{45\pi}{2}$.

2. Probar que $\iiint_V ydxdydz = 3\int_0^{\pi/2}\left(\cos^4\theta+18\operatorname{sen}^2\theta\cos^2\theta+81\operatorname{sen}^4\theta\right)dz$

3. Establecer la posición del centro de gravedad de $[V]$ [11 PUNTOS]

Sugerencia: Puede aplicarse la propiedad de las integrales eulerianas:

$$\int_0^{\pi/2}\operatorname{sen}^{(2m-1)}t\cos^{(2n-1)}tdt=\frac{B(m,n)}{2}$$

Solución

1.- Cálculo del Volumen del cuerpo $[V]$

$$V_C=\iiint_C dxdydz=\iiint_C\rho\,d\rho\,d\theta\,dy=\iint_D\rho\,d\rho\,d\theta\int_0^{\frac{x^2+9z^2}{9}}dy=\frac{1}{9}\cdot\iint_D(x^2+9z^2)\rho\,d\rho\,d\theta=$$

$$V_C=\frac{1}{9}\cdot\iint_D(\cos^2\theta+9\operatorname{sen}^2\theta)\rho^3\,d\rho\,d\theta=\frac{1}{9}\cdot\int_0^{2\pi}(1+8\operatorname{sen}^2\theta)d\theta\int_0^3\rho^3\,d\rho=$$

$$V_C=\frac{1}{9}\cdot\int_0^{2\pi}\left[1+\frac{8}{2}(1-\cos2\theta)\right]d\theta\cdot\left[\frac{\rho^4}{4}\right]\Bigg|_0^3=\frac{81}{9\cdot4}\cdot\left[5\theta-2\operatorname{sen}2\theta\right]\Big|_0^{2\pi}=\frac{45}{2}\pi\quad(u^3)$$

---

2. Probar que: $\iiint_V ydxdydz=3\int_0^{\pi/2}\left(\cos^4\theta+18\operatorname{sen}^2\theta\cos^2\theta+81\operatorname{sen}^4\theta\right)dz$

$$M_{XOZ}=\iiint_C ydxdydz=\iint_D dx\,dz\int_0^{\frac{x^2+9z^2}{9}}ydy=\frac{1}{9^2\cdot2}\cdot\iint_D\left[(x^2+9z^2)^2\right]dx\,dz=$$

$$M_{XOZ}=\frac{1}{162}\iint_D(\cos^2\theta+9\operatorname{sen}^2\theta)^2\rho^4\,\rho\,d\rho\,d\theta=\frac{1}{162}\int_0^{2\pi}(\cos^2\theta+9\operatorname{sen}^2\theta)^2d\theta\int_0^3\rho^5\,d\rho=$$

$$M_{XOZ}=\frac{1}{162}\int_0^{2\pi}(\cos^2\theta+9\operatorname{sen}^2\theta)^2d\theta\cdot\left[\frac{\rho^6}{6}\right]\Bigg|_0^3=\frac{81\cdot9}{162\cdot6}\int_0^{2\pi}(\cos^2\theta+9\operatorname{sen}^2\theta)^2d\theta=$$

$$M_{XOZ}=\frac{3}{4}\cdot4\int_0^{\pi/2}\left(\cos^4\theta+18\operatorname{sen}^2\theta\cos^2\theta+81\operatorname{sen}^4\theta\right)d\theta=$$

$$M_{XOZ}=3\int_0^{\pi/2}\left(\cos^4\theta+18\operatorname{sen}^2\theta\cos^2\theta+81\operatorname{sen}^4\theta\right)d\theta$$

---

3. **Centro de gravedad.- Cálculo**

$$M_{XOZ} = 3\int_0^{\pi/2} \cos^4\theta\, d\theta + 3\cdot 18\int_0^{\pi/2} \operatorname{sen}^2\theta \cos^2\theta\, d\theta + 3\cdot 81\int_0^{\pi/2} \operatorname{sen}^4\theta d\theta = 3\ I_1 + 54\ I_2 + 243\ I_3$$

$$M_{XOZ} = 3\cdot\frac{1}{2}B(5/2,1/2) + 54\cdot\frac{1}{2}B(3/2,3/2) + 243\cdot\frac{1}{2}B(1/2,5/2) =$$

$$M_{XOZ} = \frac{3}{2}\frac{\Gamma(5/2)\cdot\Gamma(1/2)}{\Gamma(3)} + \frac{54}{2}\frac{\Gamma(3/2)\cdot\Gamma(3/2)}{\Gamma(3)} + \frac{243}{2}\frac{\Gamma(1/2)\cdot\Gamma(5/2)}{\Gamma(3)} =$$

$$M_{XOZ} = \frac{3}{2}\frac{3/2\cdot 1/2\left(\sqrt{\pi}\right)^2}{2} + 27\frac{1/2\cdot 1/2\left(\sqrt{\pi}\right)^2}{2} + \frac{243}{2}\frac{3/2\cdot 1/2\left(\sqrt{\pi}\right)^2}{2} =$$

$$M_{XOZ} = \frac{9}{16}\pi + \frac{27}{8}\pi + \frac{729}{16}\pi = \frac{792}{16}\pi = \frac{99}{2}\pi$$

Por la simetría del cuerpo $[V]$ alrededor del eje OY, su centro de gravedad estará situado en el antedicho eje OY. Por tanto:

$$y_G = \frac{M_{XOZ}}{V_V} = \frac{99\pi/2}{45\pi/2} = \frac{99}{45} = \frac{11}{5} \qquad ; \qquad G_V = \left(0, \frac{11}{5}, 0\right)$$

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LEHEN ATALA

A.1.-Froga ezazu **zirkunferentzia bat** dela ondoko baldintza betetzen duten plano konplexuko puntuetako toki geometrikoa:

$$\arg\left(\frac{z+2}{z-2}\right)=\frac{\pi}{4} \quad (z=x+yi)\,,$$

Zirkunferentziaren zentru: $C(0,-2)$ eta erradioa: $r=2\sqrt{2}$ [6 PUNTU]

A.2.- Bira P eta Q puntuak, $x^2+y^2-12a^2=0$ zirkunferentziaren eta $y^2-4ax=0$ parabolaren arteko elkarguneak. Aipatutako edozein puntuan lerro biekiko zuzen ukitzaileek eta normalek OX ardatzarekin duten 4 elkargune batetik, eta hartutako puntutik (P edo Q) pasatzen den zuzen bertikalaren elkargunea OX -ekin bestetik, bost puntu ditugu. Puntu hauen arteko distantzia berdina dela frogatu. [7 PUNTU]

A.3.- Lehen laurdeneko erdikariaren norantzako $z=3x^4-xy+y^3$ funtzioaren deribatua $M(1,2)$ puntuan kalkulatu. [6 PUNTU]

BIGARREN ATALA

B.1.- $z=y\cdot\varphi\left(x^2-y^2\right)$ funtzioa izanik kalkulatu $E\equiv y\dfrac{\partial^2 z}{\partial x^2}+x\dfrac{\partial^2 z}{\partial x\partial y}$ adierazpenaren balioa. Funtzio arbitrario deribagarria da φ funtzioa. [6 PUNTU]

B.2.- Biz ondoko baldintza beteko duen planoko kurba-sorta: edozein $P(x,y)$ puntuan kurbarekiko normalak ordenatu-ardatzarekin duen elkargune Q izanik, Q eta P puntuetatik sistemaren jatorriarekiko distantziak berdinak dira. Kurba-sortaren ekuazio diferentziala ondokoa dela frogatu, $xdx+\left(y-\sqrt{x^2+y^2}\right)dy=0$, bere ibilbide ortogonalak kalkulatuz.

[6 PUNTU]

B.3.- Hurrengo hastapen-baldintzatako ekuazio linealetako sistemaren soluzioaren $x(t)$ edota $y(t)$, koordenatu bat, kalkulatu:

$$\begin{cases} x''(t) + y(t) = 4 \\ y''(t) + x(t) = 0 \\ x(0) = x'(0) = y(0) = y'(0) = 0 \end{cases}$$ [7 PUNTU]

HIRUGARREN ATALA

C.1.- $y^2 = \dfrac{a^2 x}{a - x}$; $2x + y = 2a$; $y = 0$ lerroek lehen koadrantean mugatzen duten D eremuaren azalera, A , honako hau dela frogatu: $A = \dfrac{a^2(\pi - 1)}{4}$.

D eremuak OX ardatz inguru biratzean sortutako gorputzaren bolumena, V, honako balioa duela frogatu baita: $V = \dfrac{3\ln 2 - 1}{3}\pi a^3$.

Oharra: $M(\frac{a}{2}, a)$ puntua da lehen bi lerroen elkargune. [11 PUNTU]

C.2.- Ondoko gainazalek mugatutakoa da, $y \geq 0$ erdiespazioko $[V]$ gorputza:

$$[\sigma_c]: \quad x^2 + z^2 = 9; \quad [\sigma_p]: \quad x^2 + 9\left(z^2 - y\right) = 0 \quad [\sigma_3]: \quad y = 0.$$

---

1. Froga ezazu $V = \dfrac{45\pi}{2}$ bere bolumena dela.

2. Frogatu: $\iiint_V y dxdydz = 3\int_0^{\pi/2} \left(\cos^4\theta + 18\sin^2\theta\cos^2\theta + 81\sin^4\theta\right) dz$.

3. $[V]$-ren grabitate-zentrua kalkulatu. [11 PUNTU]

Oharra: Integral euleriarren ondoko propietate da erabilgarri,

$$\int_0^{\pi/2} \sin^{(2m-1)} t \cos^{(2n-1)} t dt = \frac{B(m, n)}{2}$$

JARRAIBIDEAK

- Azterketaren iraupena : 3 ordubete eta erdi.

- Atalak bananduta aurkeztuko dira, ondorengo ordenean:
LEHEN Atala: azterketa hasi eta **ordubete eta 10 minutura**, **2 ordubete eta 20 minutura BIGARRENA** eta **3 ordubete eta erdira HIRUGARRENA (**azterketa amaieran).

- Kalifikazioen publikazioa: Irailaren 13an, eguerdiko 13:30-etan. (**5***n. Solairuko iragarki-oholean*).

- Berrikusketa: Irailaren 15an, eguerdiko 12etan. (**5***n. solairuko* **Laburategi Matem.**).

Bilbon, 2005.ko irailak 7

C.1 Mediante sustituciones adecuadas **reducir a eulerianas** y verificar los resultados que se indican para **dos de las tres** integrales siguientes: [10 puntos]

$$I_1 = \int_0^\infty \frac{dx}{\left(x^2+4\right)^3} = \frac{3\pi}{512};\quad I_2 = \int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} = \frac{\Gamma\left(1-1/n\right)\Gamma\left(1/n\right)}{n};\quad I_3 = \int_0^1 x\ln^5\left(1/x\right)dx = \frac{15}{8}$$

$$I_1 = \int_0^\infty \frac{dx}{\left(x^2+4\right)^3} \to \left| \begin{array}{l} x = 2\,tg\mathrm{t} \\ dx = \dfrac{2}{\cos^2 t}\ d\mathrm{t} \end{array} \right| \to I_1 = \int_0^{\pi/2} \frac{1}{\left(4\,tg^2\mathrm{t}+4\right)^3}\frac{2}{\cos^2 t}\ d\mathrm{t} =$$

$$I_1 = \frac{2}{4^3}\int_0^{\pi/2}\cos^4 t\ d\mathrm{t} = \frac{1}{32}\cdot\frac{1}{2}\cdot\beta\left(\frac{5}{2},\frac{1}{2}\right) = \frac{1}{64}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3\right)} =$$

$$I_1 = \frac{1}{64}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2} = \frac{3\pi}{512}$$

$$I_2 = \int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} \to \left| \begin{array}{l} x^n = \mathrm{t} \\ x = \mathrm{t}^{1/n} \\ \dfrac{dx}{n} = \dfrac{1}{n}\mathrm{t}^{(1/n)-1}\ d\mathrm{t} \end{array} \right| \to I_2 = \frac{1}{n}\int_0^1 (1-t)^{-1/n}t^{(1/n)-1}\ d\mathrm{t} = \frac{1}{n}\cdot\beta\left(1-\frac{1}{n},\frac{1}{n}\right)$$

por tanto: $$I_2 = \frac{1}{n}\cdot\frac{\Gamma\left(1-1/n\right)\Gamma\left(1/n\right)}{\Gamma\left(1\right)} = \frac{\Gamma\left(1-1/n\right)\Gamma\left(1/n\right)}{n}$$

$$I_3 = \int_0^1 x\ln^5\left(1/x\right)dx \to \left| \begin{array}{l} x = e^{-t} \\ dx = -e^{-t}dt \end{array} \right| \to I_3 = -\int_\infty^0 e^{-t}t^5e^{-t}dt = \int_0^\infty e^{-2t}t^5\,d\mathrm{t} =$$

$$I_3 = \int_0^\infty e^{-2t}t^5\,d\mathrm{t} \to \left| \begin{array}{l} 2t = z \\ dt = \dfrac{1}{2}dz \end{array} \right| \to I_3 = \frac{1}{2}\int_0^\infty e^{-z}\left(\frac{z}{2}\right)^5 dz = \frac{1}{2^6}\int_0^\infty z^5e^{-z}dz = \frac{1}{2^6}\Gamma(6)$$

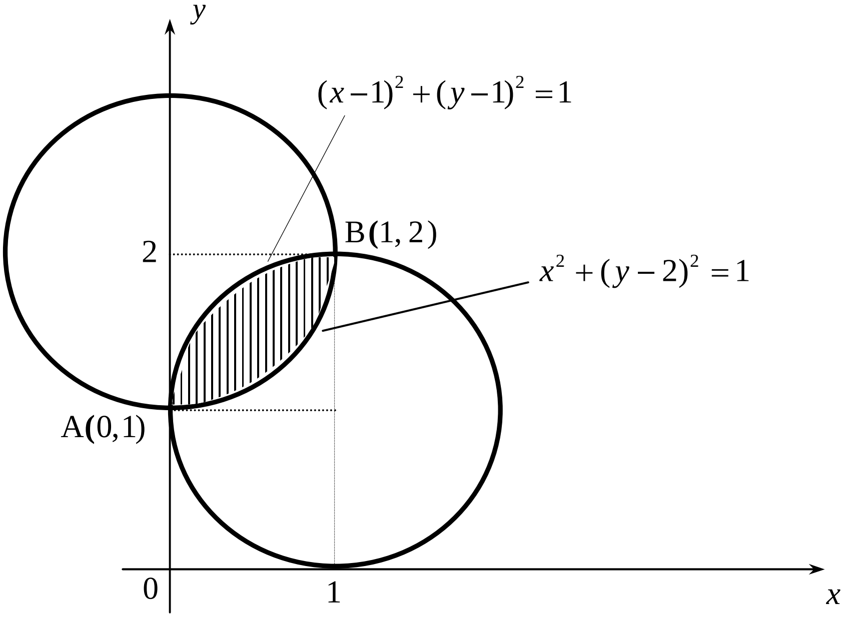
$$I_3 = \frac{1}{2^6}\Gamma(6) = \frac{1}{2^6}\cdot 5! = \frac{5\cdot4\cdot3\cdot2}{2^6} = \frac{15}{8}$$

C.2 Evaluar mediante cálculo integral el área plana definida mediante:

$[L_1]\quad x^2+y^2-2x-2y+1\le 0;\quad [L_2]\quad x^2+y^2-4y+3\le 0$ [10 puntos]

La frontera del dominio está formada por dos circunferencias:

$$\begin{cases} L_1: & (x-1)^2+(y-1)^2=1 \\ L_2: & x^2+(y-2)^2=1 \end{cases} \rightarrow \begin{cases} centro:(1,1);\ \mathrm{R}=1 \\ centro:(0,2);\ \mathrm{R}=1 \end{cases} \rightarrow \begin{cases} y=1+\sqrt{1-(x-1)^2} \\ y=2-\sqrt{1-x^2} \end{cases}$$



Cálculo del Área del dominio

$$A=\int_0^1\left[1+\sqrt{1-(x-1)^2}-(2-\sqrt{1-x^2})\right]dx=\int_0^1\left[\sqrt{1-(x-1)^2}+\sqrt{1-x^2}-1\right]dx$$

$$A=\int_0^1\sqrt{1-(x-1)^2}\ dx+\int_0^1\sqrt{1-x^2}\ dx-\int_0^1\ dx=I_1+I_2-1$$

$$I_1=\int_0^1\sqrt{1-(x-1)^2}\ dx\rightarrow\left|\begin{array}{l} x-1=\sin t\rightarrow dx=\cos t\,dt \\ \begin{cases} x=1\rightarrow t=0 \\ x=0\rightarrow t=\dfrac{-\pi}{2} \end{cases}\end{array}\right|\rightarrow I_1=\int_{-\pi/2}^0\cos^2 t\,dt=$$

$$I_1=\int_0^{\pi/2}\cos^2 t\,dt=\frac{1}{2}\cdot\beta\left(\frac{3}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2\cdot\Gamma(2)}=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2}=\frac{\pi}{4}$$

$$I_2 = \int_0^1 \sqrt{1-x^2}\, dx \rightarrow \left| \begin{array}{l} x = \sin t \rightarrow dx = \cos t\, dt \\ \begin{cases} x = 1 \rightarrow t = \dfrac{\pi}{2} \\ x = 0 \rightarrow t = 0 \end{cases} \end{array} \right| \rightarrow I_2 = \int_0^{\pi/2} \cos^2 t\, dt = \frac{\pi}{4}$$

Finalmente: $$A = I_1 + I_2 - 1 = 2\frac{\pi}{4} - 1 = \frac{\pi}{2} - 1$$

Oinarri Matematikoak I – Azterketa – 1. Deialdia
Elektrizitate eta Elektronika 2006-05-29

AZTERKETA FINALA

LEHEN ATALA

A.1.- y funtzioaren (x) aldagaiaren berredura garapena kalkulatu, x^3 duen gaian garapena mugatuz: [10 puntu]

$$y = Ln(\cos x + \sin x) \quad \rightarrow \quad y(0) = Ln(\cos 0 + \sin 0) = 0$$

$$y' = \frac{\cos x - \sin x}{\cos x + \sin x} = 1 - \frac{2\sin x}{\cos x + \sin x} \quad \rightarrow \quad y'(0) = 1 - \frac{2\sin 0}{\cos 0 + \sin 0} = 1$$

$$y'' = \frac{-2\cos x(\cos x + \sin x) - (-2\sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} = \frac{-2}{(\cos x + \sin x)^2} \quad \rightarrow$$

$$\rightarrow \quad y''(0) = \frac{-2}{(\cos 0 + \sin 0)^2} = -2$$

$$y''' = \frac{4(\cos x + \sin x)}{(\cos x + \sin x)^3} \quad \rightarrow \quad y'''(0) = \frac{4(\cos 0 + \sin 0)}{(\cos 0 + \sin 0)^3} = 4$$

Mc Laurin:

$$y(x) \cong y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3$$

$$y(x) \cong x - x^2 + \frac{2}{3}x^3$$

A.2.- Ondorengo kurbaren zuzen ukitzailearen ekuazioa $M(0,-1)$ puntuan kalkulatu

$$x^3 + 2xy + y^2 - 1 = 0 \ .$$

Zuzen horrek $N(\ )(1,-2)$ puntuan kurbarekin elkargunea duela egiaztatu.

[8 puntu]

$y = y(x)$ funtzio inplizitua deribatuz :

$$x^3 + 2xy + y^2 - 1 = 0 \quad \rightarrow \quad 3x^2 + 2y + 2xy' + 2yy' = 0$$

$$(2x + 2y)\, y' = -(3x^2 + 2y) \quad \rightarrow \quad y' = \frac{-(3x^2 + 2y)}{2x + 2y}$$

Deribatuaren balioa $M(0,-1)$ puntuan :

$$y'(0) = \left[\frac{-(3x^2 + 2y)}{2x + 2y}\right]_{(0,-1)} = \frac{-\left[3\cdot 0^2 + 2\cdot(-1)\right]}{\left[2\cdot 0 + 2\cdot(-1)\right]} = \frac{2}{-2} = -1$$

Ukitzailearen malda $M(0,-1)$ puntuan deribatuaren balioa da : $y'(0) = -1 = m_{uk}$

Ukitzailearen ekuazioa, beraz :

$y'(0) = -1 = m_{uk} \quad ; \quad M(0,-1)$

$$y - y_0 = m_{uk}(y - y_0) \rightarrow y - (-1) = (-1)(x - 0) \rightarrow \boxed{y + x = -1} \Leftrightarrow \boxed{y = -x - 1}$$

<u>Zuzen horrek $M(1,-2)$ puntuan kurbarekin elkargunea duela egiaztatu:</u>

I) $y = -x - 1$: $\rightarrow$ $y(1) = \left(-x - 1\right)_{x=1} = -2$ $\Rightarrow$ $N(1,-2) \in$ ukitzaile

$$x^3 + 2xy + y^2 - 1 = 0 \rightarrow x = 1 \rightarrow 2y + y^2 = 0 \rightarrow$$

$$\rightarrow y(2 + y) = 0 \rightarrow \begin{cases} y = 0 & \rightarrow M(1,0) \\ y = -2 & \rightarrow \boxed{N(1,-2) \in \text{kurba}} \end{cases}$$

<u>BIGARREN ATALA</u>

B.1.- Izan bedi $z = z(x, y)$ funtzioa, ondoko eran definituta:

$$L_n z = r - x \quad \text{eta :} \quad r^2 = x^2 - y^2.$$

Froga ezazu: $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{(ry^2 - x^2)z}{r^3}$. [8 puntu]

$$\text{(I)} \quad L_n z = r - x \qquad ; \qquad \text{(II)} \quad r^2 = x^2 - y^2$$

$$\text{(I)'} \quad \frac{1}{z}\cdot\frac{\partial z}{\partial y} = \frac{\partial r}{\partial y} \qquad \wedge \qquad \text{(II)} \quad 2r\cdot\frac{\partial r}{\partial y} = -2y$$

$$\text{(I)'} \quad \frac{\partial z}{\partial y} = z\cdot\frac{\partial r}{\partial y} \qquad \wedge \qquad \text{(II)'} \quad \frac{\partial r}{\partial y} = \frac{-y}{r} \quad \to \quad \text{(III)}: \quad \frac{\partial z}{\partial y} = \frac{-z\,y}{r}$$

$$\text{(III)'}: \quad \frac{\partial^2 z}{\partial y^2} = \frac{-\partial z}{\partial y}\cdot\frac{y}{r} - \frac{z}{r} - \frac{-z\,y}{r^2}\cdot\frac{\partial r}{\partial y} = \frac{z\,y}{r}\cdot\frac{y}{r} - \frac{z}{r} + \frac{z\,y}{r^2}\cdot\left(\frac{-y}{r}\right) = \frac{z\,y^2}{r^2} - \frac{z}{r} - \frac{z\,y^2}{r^3}$$

$$\text{(III)'}: \quad \frac{\partial^2 z}{\partial y^2} = \frac{z\,y^2}{r^2} - \frac{z}{r} - \frac{z\,y^2}{r^3} = \frac{z\,(r\,y^2 - r^2 - y^2)}{r^3} = \frac{z\,(r\,y^2 - x^2)}{r^3} \qquad \text{f.n.b.}$$

B.2.- Izan bedi $y''(t) + 3y'(t) + 2y(t) = f(t)$, ekuazio diferentziala.
Konboluzio-teorema erabiliz, $y(0) = y'(0) = 0$ hastapen baldintzatako ekuazio diferentzialaren soluzioaren formula integral bat aurkitu.

[8 puntu]

$$\mathfrak{L}\left[y''(t) + 3y'(t) + 2y(t)\right] = \mathfrak{L}\left[f(t)\right] \quad ; \quad y(0) = y'(0) = 0$$

$$\mathrm{p}^2\cdot \mathrm{Y(p)} + 3\mathrm{p}\cdot \mathrm{Y(p)} + 2\cdot \mathrm{Y(p)} = \mathrm{F(p)}$$

$$\mathrm{Y(p)} = \frac{\mathrm{F(p)}}{\mathrm{p}^2 + 3\mathrm{p} + 2} = \frac{\mathrm{F(p)}}{(\mathrm{p}+1)(\mathrm{p}+2)} = \frac{1}{(\mathrm{p}+3)(\mathrm{p}-2)}\cdot \mathrm{F(p)}$$

$$\boldsymbol{y(t)} = \mathfrak{L}^{-1}\left[\mathrm{Y(p)}\right] = \mathfrak{L}^{-1}\left[\frac{\mathrm{F(p)}}{(\mathrm{p}+1)(\mathrm{p}+2)}\right] = \mathfrak{L}^{-1}\left[\frac{1}{(\mathrm{p}+3)(\mathrm{p}-2)}\cdot \mathrm{F(p)}\right]$$

Konboluzio teorema

$$\boldsymbol{y(t)} = \mathfrak{L}^{-1}\left[\frac{1}{(\mathrm{p}+3)(\mathrm{p}-2)}\cdot \mathrm{F(p)}\right] = \int_0^{\mathbf{t}} \boldsymbol{f}(t-u)\cdot \boldsymbol{g}(u)\,\boldsymbol{d}u \text{ , } non \ \ \mathrm{g(t)} = \mathfrak{L}^{-1}\left[\frac{1}{(\mathrm{p}+1)(\mathrm{p}+2)}\right] \ \ \mathbf{(**)}$$

$$\mathbf{g(t)} = \mathfrak{L}^{-1}\left[\frac{1}{(p+1)(p+2)}\right] = \mathfrak{L}^{-1}\left[\frac{A}{p+1}+\frac{B}{p+2}\right] = \mathfrak{L}^{-1}\left[\frac{1}{p+1}-\frac{1}{p+2}\right] = \left[\mathbf{e^{-t}} - \mathbf{e^{-2t}}\right]$$

$$\boxed{\frac{1}{(p+1)(p+2)} = \frac{A}{p+1}+\frac{B}{p+2} \quad \begin{cases} 1 = A(p+2)+B(p+1) \\ \ p = -1: \quad 1 = A \\ \ p = -2: \quad 1 = -B \end{cases} \quad \begin{cases} A = 1 \\ B = -1 \end{cases}}$$

Konboluzio teorema honako eran aplikatuko da **(**)**:

Soluzio partikularren formula integrala: $\boxed{\boldsymbol{y(t)} = \int_0^t \boldsymbol{f}(t-u)\cdot(e^{-u}-e^{-2u})\,\boldsymbol{du}}$

B.3.- Integratu ondoko ekuazio diferentziala: $2xyy' = y^2 - x^2 - a^2$ [8 puntu]

1.- Ekuazio diferentzial A. zehatzgarrria da.

Forma diferentziala: $(x^2 - y^2 + a^2)\,dx + 2xy\,dy = 0$

$$\left.\begin{aligned} P = x^2 - y^2 + a^2 &\Rightarrow \frac{\partial P}{\partial y} = -2y \\ Q = 2xy \quad &\Rightarrow \quad \frac{\partial Q}{\partial x} = 2y \end{aligned}\right\} \quad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \rightarrow \quad \frac{P'_y - Q'_x}{Q} = \frac{-2y-2y}{2xy} = \frac{-2}{x}$$

Faktore Integrantea: $\mu = e^{-\int \frac{2}{x}dx} = e^{-2L_n x} = e^{L_n x^{-2}} = x^{-2} = \dfrac{1}{x^2}$

EDA-rekin biderkatuz: $\dfrac{x^2 - y^2 + a^2}{x^2}dx + \dfrac{2xy}{x^2}dy = 0$

Ondorengo <u>EDA zehatza</u> ondorioztatzen da:

$$\left[1 + \frac{a^2 - y^2}{x^2}\right]dx + \frac{2y}{x}dy = 0\,, \quad \text{non:} \quad \left\{\begin{aligned} P = 1 + \frac{a^2 - y^2}{x^2} &\Rightarrow \frac{\partial P}{\partial y} = \frac{-2y}{x^2} \\ Q = \frac{2y}{x} \quad &\Rightarrow \quad \frac{\partial Q}{\partial x} = \frac{-2y}{x^2} \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$U(x,y) = \int \frac{2y}{x}dy = \frac{y^2}{x} + f(x) \qquad (*)$$

$$\frac{\partial U}{\partial x} = P(x,y) = \left[1 + \frac{a^2 - y^2}{x^2}\right] = \frac{-y^2}{x^2} + f'(x) \Rightarrow f'(x) = \left(1 + \frac{a^2}{x^2}\right) \Rightarrow$$

$$\Rightarrow \quad \boxed{f(x)} = \int\left(1 + \frac{a^2}{x^2}\right)dx = \boxed{x - \frac{a^2}{x}}$$

Azkenik(*), Soluzio Orokorra: $U = \boxed{x + \dfrac{y^2 - a^2}{x} = C}$

Bernoulli-ren eredukoa da baita ere **(*linealgarria*) :**

$$2xyy' = y^2 - x^2 - a^2 \quad \Leftrightarrow \quad y' - \frac{y^2}{2xy} = \frac{-x^2 - a^2}{2xy} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad y' - \frac{1}{2x} y = \left(\frac{-x}{2} - \frac{a^2}{2x} \right) y^{-1} \qquad *$$

Ordezkaketa: $z = y^{1-(-1)} = y^{\,2}$ (1) $\xrightarrow{D_x}$ $z' = 2\,y\,y'$ $\rightarrow$ $y' = \frac{1}{2}\, z'\, y^{\,-1}$ (2)

$$y = z\, y^{\,-1} \quad (3)$$

(2) eta (3) EDA-an * ordezkatuz: $\dfrac{1}{2}\, z'\, y^{\,-1} - \dfrac{1}{2x} z\, y^{\,-1} = \left(\dfrac{-x}{2} - \dfrac{a^2}{2x} \right) y^{-1} \rightarrow$

$\rightarrow \quad z' - \dfrac{1}{x} z = \left(-x - \dfrac{a^2}{x} \right)$ Lineala (z *aldagaiarekiko*)

$\boxed{u = e^{\int \frac{1}{x} dx} = e^{\,\mathrm{L_n} x} = x}$ $\rightarrow$ $\boxed{z = u(v + C) = u\,v + Cu}$

$$v = \int \left(-x - \frac{a^2}{x} \right) x^{-1} \cdot dx = \int \left(-1 - \frac{a^2}{x^2} \right) dx = \left(-x + \frac{a^2}{x} \right)$$

Azkenik: $\boxed{z = x\left(-x + \dfrac{a^2}{x} + C \right) = \left(-x^2 + a^2 \right) + Cx}$

$$z = \left(-x^2 + a^2 \right) + Cx = y^2$$

Soluzio Orokorra: $C = \dfrac{y^2 + x^2 - a^2}{x} = x + \dfrac{y^2 - a^2}{x}$

HIRUGARREN ATALA

C.1.- Ondorengo hiru integraletatik, ebatzi **bi** integral:

$$I_1 = \int \frac{dx}{(x-2)\sqrt{x-1}}; \quad I_2 = \int \frac{dx}{1+tgx}; \quad I_3 = \int \left(\frac{x-1}{x+1}\right)^2 dx$$ [8 puntu]

Binomio simplea da aurrenekoa, lehen mailako binomio izanik, hau razionaltzean datza bidea:

$$\sqrt{x-1} = t \quad \to \quad x-1 = t^2 \quad \xrightarrow{d} \quad dx = 2tdt$$

$$I_1 = \int \frac{dx}{(x-2)\sqrt{x-1}} \equiv \int \frac{2tdt}{\left(t^2-1\right)t} = -2\arg tht + C = -\arg th\sqrt{x-1} + C \ \to \ I_1 = \ln\left|\frac{\sqrt{x-1}-1}{\sqrt{x-1}+1}\right| + C$$

Razionalgarria da bigarrena, aldaketa unibertsala dagokio:

$$tgx = t \to x = arctgt \quad \xrightarrow{d} \quad dx = \frac{dt}{1+t^2} \quad \Rightarrow \quad I_2 = \int \frac{dx}{1+tgx} \equiv \int \frac{dt}{(1+t)\left(1+t^2\right)}$$

Zatiki sinpleetan deskonposatuz,

$$\frac{1}{(1+t)\left(1+t^2\right)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2} = \frac{A\left(1+t^2\right)+(Bt+C)(1+t)}{(1+t)\left(1+t^2\right)}$$

$$\begin{cases} \# = 2: & 0 = A+B \\ \# = 1: & 0 = B+C \\ \# = 0: & 1 = A+C \end{cases} \quad \to \quad A = C = \frac{1}{2}; \quad B = -\frac{1}{2}$$

$$\int \frac{dt}{(1+t)\left(1+t^2\right)} = \frac{1}{2}\int\left(\frac{1}{1+t} + \frac{-t+1}{1+t^2}\right)dt = \frac{1}{2}\left[\ln(1+t) - \frac{1}{2}\ln\left(1+t^2\right) + artgt\right] \equiv$$

$$\equiv \frac{1}{2}\left[\ln(1+tgx) - \frac{1}{2}\ln\left(1+tg^2x\right) + x\right] = \frac{1}{2}\left[\ln\left|(1+tgx)cosx\right| + x\right] \quad \to$$

$$I_2 = \int \frac{dx}{1+tgx} = \frac{x + \ln\left|\sin x + \cos x\right|}{2} + C$$

Maila bereko polinomioak izanik, zatidura kalkulatuko dugu:

$$I_3 = \int \left(\frac{x-1}{x+1}\right)^2 dx = \int \left(1 - \frac{2}{x+1}\right)^2 dx = \int \left(1 - \frac{4}{x+1} + \frac{4}{(x+1)^2}\right)dx = x - 4\ln|x+1| - \frac{4}{x+1} + C$$

C.2- Izan bedi **[*V*],** ondorengo gainazalek mugatutako gorputz homogenoa:

$$[\sigma_1]:\quad x^2+y^2=9;\quad [\sigma_2]:\quad x+z-6=0;\quad [\sigma_3]:\quad x+3z-6=0$$

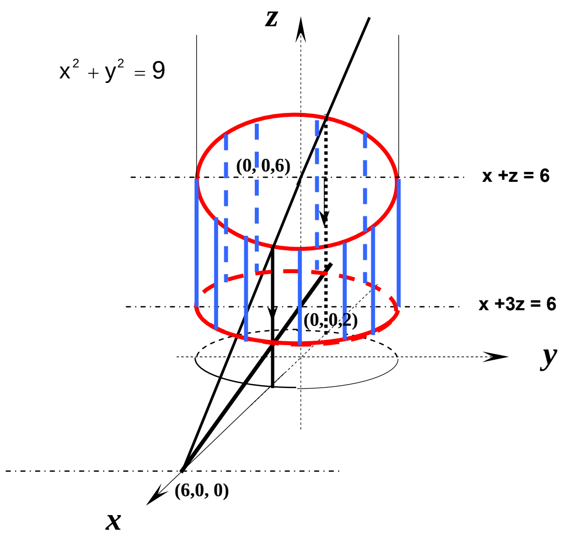
1.- **[*V*]** gorputza marraztu.

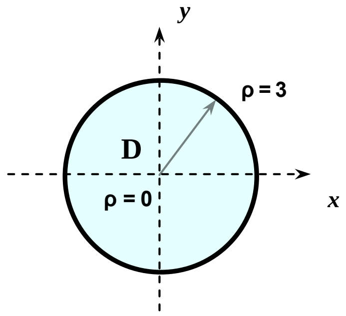
2.-Gorputzaren bolumena $V_V = 36\pi$ dela egiaztatu, eta bere grabitate-zentruaren z_c koordenatua $z_c = \dfrac{17}{4}$ egiaztatu baita ere. [10 puntu]

σ_1 **:** $x^2+y^2-9=0$ **Gainazal zilindriko zirkularra ;** sortzaileak // OZ

Gorputzaren proiekzio egokia: XOY planokoa: $x^2+y^2=9$ zirkunferentzia $\Leftrightarrow$ $\rho = 2$
Integrazio-ordena egokia: eskuineko integrala **z** aldagaiarekiko.

$$\begin{cases} x+z-6=0 \ \ (\textit{plano proiektante // OX}) \Rightarrow \quad z=6-x \quad \rightarrow \boxed{z=6-\rho\cos\theta} \quad \textit{G.Maiorante} \\ x+3z-6=0 \ \ (\textit{plano proiektante // OX}) \Rightarrow \quad z=(6-x)/3 \rightarrow \boxed{z=(6-\rho\cos\theta)/3} \quad \textit{G.Minorante} \end{cases}$$





V gorputzaren Proiekzio zilindrikoa XOY planoan

Bolumena:

$$V_C = \iiint_C dx\,dy\,dz = \iiint_C \rho\,d\rho\,d\theta\,dz =$$

$$V_V = \iint_{D_H} \rho\,d\theta\,d\rho \int_{(6-\rho\cos\theta)/3}^{6-\rho\cos\theta} dz = \frac{2}{3}\int_0^{2\pi} d\theta \int_0^3 (6-\rho\cos\theta)\,\rho\,d\rho = \frac{2}{3}\int_0^{2\pi} \left[6\frac{\rho^2}{2} - \frac{\rho^3}{3}\cos\theta \right]_0^3 d\theta =$$

$$V_V = \frac{2}{3}\int_0^{2\pi} [27-9\cos\theta-(0-0)]\,d\theta = \frac{2}{3}\left[\left(27\theta - 9\sin\theta\right)\Big|_0^{2\pi} \right] = \frac{2}{3}\left[54\pi + 9\sin 2\pi - \left(0+\frac{8}{3}\sin 0\right) \right] = \boxed{36\pi\ u^3}$$

XOZ planoarekiko momentua $\mathbf{M_{XOY}} = \iiint_C \mathbf{z\,dx\,dy\,dz} = \iiint_C \boldsymbol{\rho\,d\rho\,d\theta\,z\,dz} =$

$$\mathbf{M_{XOY}}/\delta = \iint_D \rho\,d\theta\,d\rho \int_{(6-\rho\cos\theta)/3}^{6-\rho\cos\theta} z\,dz = \iint_D \left[z^2/2\right]\Big|_{(6-\rho\cos\theta)/3}^{6-\rho\cos\theta} \rho\,d\theta\,d\rho =$$

$$\mathbf{M_{XOY}}/\delta = \frac{1}{2}\cdot\left(1-\frac{1}{3^2}\right)\int_0^{2\pi} d\theta \int_0^3 (6-\rho\cos\theta)^2\,\rho\,d\rho =$$

$$\mathbf{M_{XOY}}/\delta = \frac{4}{9}\int_0^{2\pi} d\theta \int_0^3 (36\rho - 12\rho^2\cos\theta + \rho^3\cos^2\theta)\,d\rho = \frac{4}{9}\int_0^{2\pi}\left[36\frac{\rho^2}{2} - 12\frac{\rho^3}{3}\cos\theta + \frac{\rho^4}{4}\cos^2\theta\right]\Big|_0^3 d\theta =$$

$$\mathbf{M_{XOY}}/\delta = \frac{4}{9}\int_0^{2\pi}\left[162 - 108\cos\theta + \frac{81}{4\cdot 2}\,(1+\cos 2\theta)\right]d\theta = \frac{4}{9}\left[162\,\theta - 108\sin\theta + \frac{81}{8}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]\Big|_0^{2\pi} =$$

$$\mathbf{M_{XOY}}/\delta = \frac{4}{9}\left[324\pi - 108\sin 2\pi + \frac{81}{8}\left(2\pi + \frac{\sin 2\cdot 2\pi}{2}\right) - \left(0 - 108\sin 0 + \frac{81}{8}\left(0 + \frac{\sin 0}{2}\right)\right)\right] =$$

$$\mathbf{M_{XOY}}/\delta = \frac{4}{9}\left[324\pi + \frac{81}{4}\pi\right] = \frac{4}{9}\cdot\frac{1377}{4}\pi = \frac{1377}{9}\pi = \boxed{153\pi}$$

Beraz:

$$z_G = \frac{\mathbf{M_{XOY}}/\delta}{m/\delta} = \frac{\mathbf{M_{XOY}}/\delta}{V_C} = \frac{153\pi}{36\pi} = \boxed{\frac{17}{4}}$$

Oinarri Matematikoak I – Azterketa – 1. Deialdia
Elektrizitate eta Elektronika 2006-05-29

AZTERKETA FINALA

LEHEN ATALA

A.1.- y funtzioaren (x) aldagaiaren berredura garapena kalkulatu, x^3 duen gaian garapena mugatuz:

$$y = Ln(\cos x + \sin x)$$

[10 puntu]

A.2.- Ondorengo kurbaren zuzen ukitzailearen ekuazioa $M(0,-1)$ puntuan kalkulatu

$$x^3 + 2xy + y^2 - 1 = 0 \;.$$

Zuzen horrek $M(1,-2)$ puntuan kurbarekin elkargunea duela egiaztatu.

[8 puntu]

BIGARREN ATALA

B.1.- Izan bedi $z = z(x, y)$ funtzioa, ondoko eran definituta:

$$L_n z = r - x \quad \text{eta :} \quad r^2 = x^2 - y^2.$$

Froga ezazu: $$\frac{\partial^2 z}{\partial y^2} = \frac{(ry^2 - x^2)z}{r^3} \;.$$ [8 puntu]

B.2.- Izan bedi $y''(t) + 3y'(t) + 2y(t) = f(t)$, ekuazio diferentziala.

Konboluzio-teorema erabiliz, $y(0) = y'(0) = 0$ hastapen baldintzatako ekuazio diferentzialaren soluzio partikularren formula integral bat aurkitu.

[8 puntu]

B.3.- Integratu ondoko ekuazio diferentziala: $2xyy' = y^2 - x^2 - a^2$ [8 puntu]

HIRUGARREN ATALA

C.1.- Ondorengo hiru integraletatik, ebatzi **bi** integral:

$$I_1 = \int \frac{dx}{(x-2)\sqrt{x-1}}; \quad I_2 = \int \frac{dx}{1+tgx}; \quad I_3 = \int \left(\frac{x-1}{x+1}\right)^2 dx$$ [8 puntu]

C.2- Izan bedi **[*V*],** ondorengo gainazalek mugatutako gorputz homogenoa:

$$[\sigma_1]: \quad x^2 + y^2 = 9; \quad [\sigma_2]: \quad x + z - 6 = 0; \quad [\sigma_3]: \quad x + 3z - 6 = 0$$

1.- **[*V*]** gorputza marraztu.

2.-Gorputzaren bolumena $V_V = 36\pi$ dela egiaztatu, eta bere grabitate-zentruaren z_c koordenatua $z_c = \frac{17}{4}$ egiaztatu baita ere. [10 puntu]

JARRAIBIDEAK

- Mota guztietako bibliografia erabilgarria da.

- **Azterketaren iraupena :** 3 ordubete eta 15 minutu.

- **Atalak bananduta** aurkeztuko dira, ondorengo ordenean:
  - **Lehen Atala:** azterketa hasi eta **ORDUBETERA.**
  - **Bigarren Atala:** azterketa hasi eta **BI ORDUBETERA.**
  - **Hirugarren Atala:** azterketa hasi eta **HIRU ORDUBETE eta Laurdenera.**

- **Ariketen puntuaketa** (60 puntu, guztira)

- **Kalifikazioen publikazioa:** Ekainaren 7an, eguerdiko 13:30-etan.
  (**5***n. Solairuko iragarki-oholean*).

- **Berrikusketa:** Ekainaren 14an, eguerdiko 12etan.
  (**5***n. solairuko* **Laburategi matematikoan**).

Bilbon, 2006.eko maiatzak 29

MATEMATIKAREN HEDAPENA - AZTERKETA FINALA - Bilbo, 03-09-01
Injeniaritza Elektrikoa eta Elektronikoa

[**A orria**]

Kalkula ezazu ondoko ekuazio diferentzialaren soluzio orokorra,

$$\mathbf{y'' + 2y' + 5y = e^{-x}\cos 2x,}$$

ondoren adierazitako bi **metodo erabiliz**:
1) Koefiziente indeterminatuen metodoa. 2) Parametroen aldakuntza metodoa.

Ekuazio karakteristikoa: $r^2 + 2r + 5 = 0 \quad \Rightarrow \quad r = -1 \pm 2i \;\; (s = 1) \begin{cases} y_1 = e^{-x}\cos 2x \\ y_2 = e^{-x}\sin 2x \end{cases}$

Homogeno asoziatuaren Soluzio orokorra: $\boxed{y_{HA} = C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x}$

1) Koefiziente indeterminatuen metodoa.

$$\underbrace{e^{-x}\cos 2x}_{gai\ ez-homogenoa} \Rightarrow \quad Y_O = e^{-x}\left(A\cos 2x + B\sin 2x\right)\cdot x$$

$$Y_0' = e^{-x}(-x+1)\left(A\cos 2x + B\sin 2x\right) + xe^{-x}\left(-2A\sin 2x + 2B\cos 2x\right)$$

$$Y_0' = xe^{-x}\left[(-A+2B)\cos 2x - (2A+B)\sin 2x\right] + e^{-x}\left(A\cos 2x + B\sin 2x\right)$$

$$\begin{aligned} Y_0'' = {} & e^{-x}(-x+1)\left[(-A+2B)\cos 2x - (2A+B)\sin 2x\right] + \\ & + xe^{-x}\left[(-4A-2B)\cos 2x + (2A-4B)\sin 2x\right] + e^{-x}\left[(-A+2B)\cos 2x - (2A+B)\sin 2x\right] \end{aligned}$$

$$Y_0'' = xe^{-x}\left[(-3A-4B)\cos 2x + (4A-3B)\sin 2x\right] + e^{-x}\left[(-2A+4B)\cos 2x + (-4A-2B)\sin 2x\right]$$

$$Y_0'' + 2Y_0' + 5Y_0 = xe^{-x}\left[(-3A-4B-2A+4B+5A)\cos 2x + (4A-3B-4A-2B+5B)\sin 2x\right] +$$

$$+e^{-x}\left[(-2A+4B+2A)\cos 2x + (-4A-2B+2B)\sin 2x\right] =$$

$$Y_0'' + 2Y_0' + 5Y_0 = xe^{-x}\left[0\cdot\cos 2x + 0\cdot\sin 2x\right] + \underbrace{e^{-x}\left[4B\cos 2x - 4A\sin 2x\right] = e^{-x}\cos 2x}_{koefizienteak\ berdinduz}$$

Azkenik: **A = 0, B = ¼**

Ekuazio osotuaren Soluzio orokorra: $\boxed{y = y_{HA} + \mathbf{Y_O} = \mathbf{e^{-x}\left[C_1 \cos 2x + \left(C_2 + \frac{x}{4}\right)\sin 2x\right]}}$

2) <u>Parametroen aldakuntza metodoa.</u> $y = L_1 e^{-x}\cos 2x + L_2 e^{-x}\sin 2x$

$$\begin{cases} L_1' e^{-x}\cos 2x \quad + \quad L_2' e^{-x}\sin 2x \quad = \quad 0 \\ L_1' e^{-x}(-\cos 2x - 2\sin 2x) + L_2' e^{-x}(-\sin 2x + 2\cos 2x) = e^{-x}\cos 2x \end{cases}$$

$$L_1' = \frac{\begin{vmatrix} 0 & \sin 2x \\ \cos 2x & -\sin 2x + 2\cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -\cos 2x - 2\sin 2x & -\sin 2x + 2\cos 2x \end{vmatrix}} = \frac{-\sin 2x\cos 2x}{2} = \frac{-\sin 4x}{4}$$

$$L_1 = \frac{-1}{4}\int \sin 4x\,dx = \frac{1}{16}\cos 4x + C_1$$

$$L_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -\cos 2x - 2\sin 2x & \cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -\cos 2x - 2\sin 2x & -\sin 2x + 2\cos 2x \end{vmatrix}} = \frac{\cos^2 2x}{2}$$

$$L_2 = \frac{1}{2}\int \frac{1+\cos 4x}{2}dx = \frac{1}{4}\left(x + \frac{\sin 4x}{4}\right) + C_2$$

Azkenik: $y = L_1 e^{-x}\cos 2x + L_2 e^{-x}\sin 2x$

$$y = e^{-x}\cos 2x\left(\frac{\cos 4x}{16} + C_1\right) + e^{-x}\sin 2x\left(\frac{x}{4} + \frac{\sin 4x}{16} + C_2\right)$$

$$\boxed{\mathbf{y = e^{-x}\left[C_1 \cos 2x + \left(C_2 + \frac{x}{4}\right)\sin 2x\right]}} \quad \mathbf{(*)}$$

$$e^{-x}\left[\frac{\cos 4x}{16}\cdot\cos 2x+\frac{\sin 4x}{16}\sin 2x\right]=e^{-x}\,\frac{\cos(4x-2x)}{16}=e^{-x}\,\frac{\cos 2x}{16}$$

(*)

$$e^{-x}\,\frac{\cos 2x}{16}\subset y_{HA}$$

[**B orria**]

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3) Askatu ondoren adierazitako ekuazio diferentzialetako sistema,

$$\begin{cases} x'(t)+y'(t)-x(t)-y(t)+t+2=0 \\ x'(t)-y'(t)+x(t)-y(t)-t=0 \end{cases},$$

hurrengo hastapen baldintzatarako: x(0) = 1; y(0) = -1

$$\begin{cases} p\cdot X(p)-1+p\cdot Y(p)+1-X(p)-Y(p)=\dfrac{-1}{p^2}-\dfrac{2}{p} \\ p\cdot X(p)-1-p\cdot Y(p)-1+X(p)-Y(p)=\dfrac{1}{p^2} \end{cases}\qquad \begin{cases} (p-1)\cdot X(p)+(p-1)\cdot Y(p)=\dfrac{-2p-1}{p^2} \\ (p+1)\cdot X(p)-(p+1)\cdot Y(p)=\dfrac{2p^2+1}{p^2} \end{cases}$$

$$\begin{cases} X(p)+Y(p)=\dfrac{-2p-1}{p^2(p-1)} \\ X(p)-Y(p)=\dfrac{2p^2+1}{p^2(p+1)} \end{cases}\qquad \begin{cases} X(p)=\dfrac{(-2p-1)(p+1)+(2p^2+1)(p-1)}{2p^2(p+1)(p-1)}=\dfrac{p^3-2p^2-p-1}{p^2(p+1)(p-1)} \\ Y(p)=\dfrac{(-2p-1)(p+1)-(2p^2+1)(p-1)}{2p^2(p+1)(p-1)}=\dfrac{-2p^2-4}{2p(p+1)(p-1)} \end{cases}$$

$$X(p)=\frac{p^3-2p^2-p-1}{p^2(p+1)(p-1)}=\frac{A}{p}+\frac{B}{p^2}+\frac{C}{p+1}+\frac{D}{p-1}$$

$$p^3-2p^2-p-1=Ap(p+1)(p-1)+B(p+1)(p-1)+Cp^2(p-1)+Dp^2(p+1)$$

3. maila:	**1 = A + B + C**	**A = 1**
p=0:	**-1 = - B**	**B = 1**
p=1:	**-3 = 2D**	**C = 3/2**
p=-1:	**-3 = -2C**	**D = -3/2**

$$\boxed{x(t)}=\mathfrak{L}^{-1}\left[\mathbf{X(p)}\right]=\mathfrak{L}^{-1}\left[\frac{1}{p}+\frac{1}{p^2}+\frac{3/2}{p+1}-\frac{3/2}{p-1}\right]=\boxed{1+t+\frac{3}{2}(e^{-t}-e^{t})}=\boxed{\mathbf{1+t-3Sht}}$$

$$Y(p)=\frac{-p^2-2}{p(p+1)(p-1)}=\frac{A}{p}+\frac{B}{p+1}+\frac{C}{p-1}\qquad -p^2-2=A(p+1)(p-1)+Bp(p-1)+Cp(p+1)$$

p=0:	**-2 = - A**	**A = 2**
p=1:	**-3 = 2C**	**B = -3/2**
p=-1:	**-3 = 2B**	**C = -3/2**

$$\boxed{y(t)} = \mathfrak{L}^{-1}\left[\mathbf{Y(p)}\right] = \mathfrak{L}^{-1}\left[\frac{2}{p} - \frac{3/2}{p+1} - \frac{3/2}{p-1}\right] = \boxed{2 - \frac{3}{2}(e^{-t} + e^{t})} = \boxed{\mathbf{2 - 3Cht}}$$

---

4) Konboluzio teorema erabiliz, ondoren adierazitako ekuazio diferentzialaren soluzioaren formula integral bat aurkitu,

$$\mathbf{x'' + x' - 6x = f(t);\ \ x(0) = 1;\ \ x'(0) = 2}$$

Ebatz ezazu $f(t) = 25e^{-3t}$ denerako.

$$\mathfrak{L}\left[x'' + x' - 6x\right] = \mathfrak{L}\left[f(t)\right] \quad ; \quad x(0) = 1;\ \ x'(0) = 2$$

$$p^2 \cdot X(p) - p - 2 + p \cdot X(p) - 1 - 6 \cdot X(p) = F(p)$$

$$X(p) = \frac{F(p)}{p^2 + p - 6} + \frac{p+3}{p^2 + p - 6} = \frac{F(p)}{(p+3)(p-2)} + \frac{p+3}{(p+3)(p-2)} = \frac{F(p)}{(p+3)(p-2)} + \frac{1}{(p-2)}$$

$$\boldsymbol{x(t)} = \mathfrak{L}^{-1}\left[X(p)\right] = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p+3)(p-2)} + \frac{1}{(p-2)}\right] = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p+3)(p-2)}\right] + \mathfrak{L}^{-1}\left[\frac{1}{(p-2)}\right] =$$

$$\boldsymbol{x(t)} = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p+3)(p-2)}\right] + \boldsymbol{e^{2t}} = \boldsymbol{e^{2t}} + h(t) \qquad \textbf{(*)}$$

Konboluzio teorema

$$\boldsymbol{h(t)} = \mathfrak{L}^{-1}\left[F(p) \cdot \frac{1}{(p+3)(p-2)}\right] = \int_0^{\mathbf{t}} \boldsymbol{f}(t-\tau) \cdot \boldsymbol{g}(\tau)\, \boldsymbol{d}\tau \quad , \ non \quad g(t) = \mathfrak{L}^{-1}\left[\frac{1}{(p+3)(p-2)}\right]$$

$$\mathbf{g(t)} = \mathfrak{L}^{-1}\left[\frac{1}{(p+3)(p-2)}\right] = \mathfrak{L}^{-1}\left[\frac{A}{p+3} + \frac{B}{p-2}\right] = \ldots.. = \mathfrak{L}^{-1}\left[\frac{-1/5}{p+3} + \frac{1/5}{p-2}\right] = \frac{1}{5}\left[\mathbf{e^{2t}} - \mathbf{e^{-3t}}\right]$$

beraz: $$\boldsymbol{h(t)} = \mathfrak{L}^{-1}\left[\frac{F(p)}{(p+3)(p-2)}\right] = \frac{1}{5}\int_0^{\mathbf{t}} \boldsymbol{f}(t-\tau) \cdot (e^{2\tau} - e^{-3\tau})\, \boldsymbol{d}\tau$$

azkenik: ***(*)*** $$\boxed{\boldsymbol{x(t)} = \boldsymbol{e^{2t}} + \frac{1}{5}\int_0^{\mathbf{t}} \boldsymbol{f}(t-\tau) \cdot (e^{2\tau} - e^{-3\tau})\, \boldsymbol{d}\tau}$$

$f(t) = 25e^{-3t}$ denerako:

$$\boldsymbol{h(t)} = \mathfrak{L}^{-1}\left[\frac{\mathrm{F(p)}}{\mathrm{(p+3)(p-2)}}\right] = \frac{25}{5}\int_0^{t} e^{-3(t-\tau)}\cdot(e^{2\tau}-e^{-3\tau})\,d\tau = 5e^{-3t}\int_0^{t} e^{3\tau}(e^{2\tau}-e^{-3\tau})\,d\tau =$$

$$\boldsymbol{h(t)} = 5e^{-3t}\int_0^{t}(e^{5\tau}-1)\,d\tau = 5e^{-3t}\left[\frac{e^{5\tau}}{5}-t\right]\Bigg|_0^t = 5e^{-3t}\left[\frac{e^{5t}}{5}-t-\left(\frac{1}{5}-0\right)\right] =$$

$$\boxed{\boldsymbol{h(t)} = e^{2t} - 5te^{-3t} - e^{-3t}}$$

Azkenik: $\boxed{\boldsymbol{x(t)}} = h(t) + \boldsymbol{e^{2t}} = \boxed{2\boldsymbol{e^{2t}} - 5te^{-3t} - e^{-3t}}$

[**C orria**]

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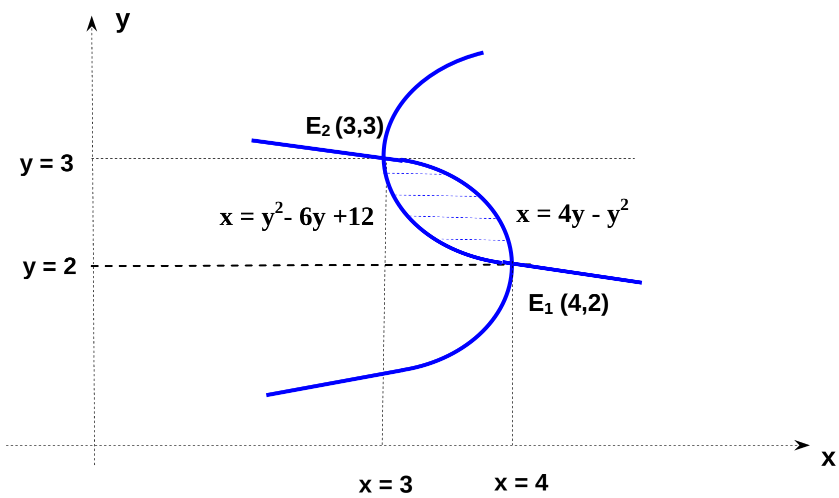
5) Ondorengo lerroek mugatutako eremu planoaren grabitate zentruaren y_C koordenatua kalkulatu:

$$\mathbf{y^2 - 4y + x = 0; \quad y^2 - 6y - x + 12 = 0}$$



AZALERA

$$\mathbf{A_D} = \iint_D dx\,dy = \int_2^3 dy\int_{y^2-6y+12}^{4y-y^2} dx = \int_2^3 (10y - 2y^2 - 12)dy = \left[10\frac{y^2}{2} - 2\frac{y^3}{3} - 12y\right]\Bigg|_2^3 =$$

$$\mathbf{A_D} = 45 - 18 - 36 - \left(20 - \frac{16}{3} - 24\right) = \frac{16}{3} - 5 = \boxed{\frac{1}{3}}\ (u^2)$$

ARDATZEKIKO MOMENTUAK

$$\mathbf{M_{OX}} = \iint_D y\,dx\,dy = \int_2^3 y\,dy\int_{y^2-6y+12}^{4y-y^2} dx = \int_2^3 (10y-2y^2-12)y\,dy = \left[10\frac{y^3}{3}-2\frac{y^4}{4}-12\frac{y^2}{2}\right]\Bigg|_2^3 =$$

$$\mathbf{M_{OX}} = 90-\frac{81}{2}-54-\left(\frac{80}{3}-8-24\right)=68-\frac{403}{6}=\boxed{\frac{5}{6}}$$

$$\text{G, } \textit{beraz}: \qquad G=\left(\frac{M_{OY}}{m},\frac{M_{OX}}{m}\right)=\left(\frac{M_{OY}}{A_D},\frac{M_{OX}}{A_D}\right)=\left(\frac{7/6}{1/3},\frac{5/6}{1/3}\right)=\boxed{\left(\frac{7}{2},\frac{5}{2}\right)}$$

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[6] Biz [C] gorputz homogenoa, ondoren adierazitako gainazalek mugatutakoa,

$$\mathbf{x^2+y^2+z^2-25=0\ \ (0\le y\le 3)\ ;\ \ x^2+z^2+y-19=0\ \ (3\le y\le 19)\ ;\ \ y=0}$$

6.1 [C] gorputzaren grabitate-zentrua kalkulatu.

6.2 [C] gorputza mugatzen duten gainazalen azalera totala kalkulatu.

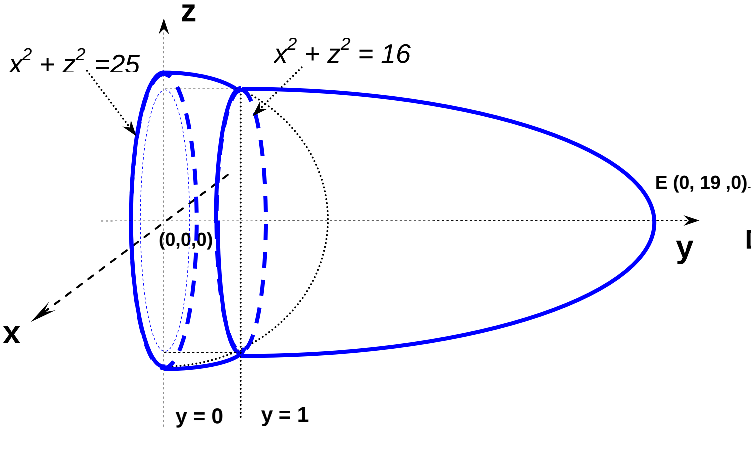
Analisia:

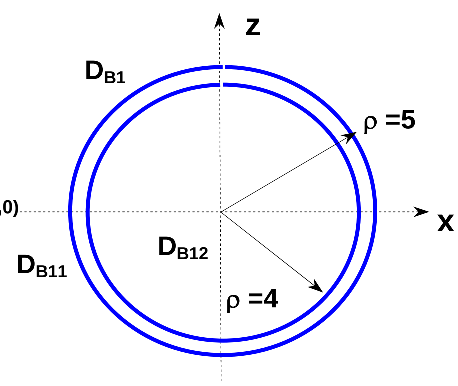
$x^2+y^2+z^2-25=0 \quad (0\le y\le 3)$: *(big. mailako hiru gai, koef. berekoak)* - $\boxed{\textbf{\textit{Esfera}}}$

$$\begin{cases}
\text{Biraketa – ardatza } (\textbf{y}\ \textit{aldagaiari dagokiona}): \boxed{OY, (x=z=0)};\ \text{zentru}: \boxed{O(0,0,0)} \\
\text{Erradioa}: \sqrt{25}=\boxed{5};\quad x=z=0 \ \Rightarrow y=5,\ \ \textit{beraz}: \boxed{P(0,5,0)} \\
\text{Eskuin - integralaren limitea } (y\ \textit{askatu}): \boxed{y=\sqrt{25-x^2-z^2}} \Leftrightarrow \boxed{y=\sqrt{25-\rho^2}} \\
\underbrace{\text{Elkarguneak } (\textit{zirkunf.})}_{\text{proiekzioak}}: \begin{cases} y=0\ \ (\textit{ekuazioan ordezk.}): \boxed{x^2+z^2=25};\ \boxed{\rho=5} \\ y=3\ \ (\textit{ekuazioan ordezk.}): \boxed{x^2+z^2=16};\ \boxed{\rho=4} \end{cases}
\end{cases}$$

$x^2+z^2+y-19=0 \quad (3\le y\le 19)$: *(big. mailako bi gai, koef. berekoak)* - $\boxed{\textbf{\textit{Paraboloide}}}$

$$\begin{cases}
\text{Biraketa – ardatza } (\textbf{y}\ \textit{aldagaiari dagokiona}): \boxed{OY, (x=z=0)};\ \text{Erpina}: \boxed{E(0,19,0)}\ (\textit{maximoa}) \\
\text{Eskuin - integralaren limitea } (y\ \textit{askatu}): \boxed{y=19-x^2-z^2} \Leftrightarrow \boxed{y=19-\rho^2} \\
\underbrace{\text{Elkarguneak } (\textit{zirkunf.})}_{\text{proiekzioak}}: \begin{cases} y=3\ \ (\textit{ekuazioan ordezk.}): \boxed{x^2+z^2=16};\ \boxed{\rho=4} \\ y=19\ \ (\textit{ekuazioan ordezk.}): \boxed{x^2+z^2=0};\ \ \rho=0 \Leftrightarrow \boxed{E(0,19,0)} \end{cases}
\end{cases}$$





Masa zentru - Kalkuluak

Bolumena

$$V_C = \iiint_C dx\,dy\,dz = \iiint_C \rho\, d\rho\, d\theta\, dy = \iint_{DB11} \rho\, d\rho\, d\theta \int_0^{\sqrt{25-\rho^2}} dy + \iint_{DB12} \rho\, d\rho\, d\theta \int_0^{19-\rho^2} dy =$$

$$V_C = \int_0^{2\pi} d\theta \int_4^5 \left[25-\rho^2\right]^{1/2} \rho\, d\rho + \int_0^{2\pi} d\theta \int_0^4 \left[19-\rho^2\right] \rho\, d\rho = 2\pi \left[\frac{\left(25-\rho^2\right)^{3/2}}{(3/2)\cdot(-2)}\right]_4^5 +$$

$$+ 2\pi \left[\frac{\left(19-\rho^2\right)^2}{2\cdot(-2)}\right]_0^4 = \frac{-2\pi}{3}\left[0 - 9^{3/2}\right] - \frac{\pi}{2}\left[3^2 - 19^2\right] = \boxed{194\pi} \quad \textit{(}u^3\textit{)}$$

M $_{XOZ}$
momentu

$$M_{XOZ} = \iiint_C y\, dx\,dy\,dz = \iiint_C \rho\, d\rho\, d\theta\, y\, dy = \iint_{DB11} \rho\, d\rho\, d\theta \int_0^{\sqrt{25-\rho^2}} y\, dy + \iint_{DB12} \rho\, d\rho\, d\theta \int_0^{19-\rho^2} y\, dy =$$

$$M_{XOZ} = \frac{1}{2}\int_0^{2\pi} d\theta \int_4^5 \left(25-\rho^2\right) \rho\, d\rho + \frac{1}{2}\int_0^{2\pi} d\theta \int_0^4 \left(19-\rho^2\right)^2 \rho\, d\rho = \frac{2\pi}{2}\left[\frac{\left(25-\rho^2\right)^2}{2\cdot(-2)}\right]_4^5 + \frac{2\pi}{2}\left[\frac{\left(19-\rho^2\right)^3}{3\cdot(-2)}\right]_0^4 =$$

$$M_{XOZ} = \frac{-\pi}{4}\left[0 - 9^2\right] - \frac{\pi}{6}\left[3^3 - 19^3\right] = \boxed{\frac{13907\pi}{12}} \quad (m \cdot u^2)$$

G, *beraz :* $\qquad G = \left(0, \frac{13907\pi/12}{194\pi}, 0\right) = \boxed{\left(0, \frac{13907}{2328}, 0\right)}$

Azalera

$$x^2 + z^2 + y - 19 = 0 \quad (3 \le y \le 19) : \boxed{\textbf{\textit{Paraboloide}}}$$

$$A_{\sigma 2} = \iint_\sigma d\sigma = \iint_\sigma \frac{dx\,dz}{|\cos\beta|} = \iint_\sigma \sqrt{(2x)^2 + (2z)^2 + 1^2}\; dx\,dz = \iint_{DB12} \sqrt{4\rho^2+1}\,\rho d\rho d\theta =$$

$$A_{\sigma 2} = \int_0^{2\pi} d\theta \int_0^4 \left(4\rho^2+1\right)^{1/2} \rho\, d\rho = 2\pi\left[\frac{\left(4\rho^2+1\right)^{3/2}}{(3/2)\cdot(8)}\right]_0^4 = \boxed{\frac{\pi}{6}\left[65^{3/2} - 1\right]} \; (u^2)$$

$$x^2 + y^2 + z^2 - 25 = 0 \quad (0 \le y \le 3) : \boxed{\textbf{\textit{Esfera}}}$$

$$A_{\sigma 1} = \iint_\sigma d\sigma = \iint_\sigma \frac{dx\,dz}{|\cos\beta|} = \iint_\sigma \frac{5}{y}\, dx\,dz = \iint_{DB11} \frac{5}{\sqrt{25-\rho^2}}\,\rho d\rho d\theta =$$

$$A_{\sigma 1} = 5\int_0^{2\pi} d\theta \int_{\sqrt{3}}^2 \left(25-\rho^2\right)^{-1/2} \rho\, d\rho = 2\pi\cdot 5\left[\frac{\left(25-\rho^2\right)^{1/2}}{(1/2)\cdot(-2)}\right]_4^5 = -10\pi\left[0 - 9^{1/2}\right] = \boxed{30\pi} \; (u^2)$$

AZALERA Totala: $A_T = A_{\sigma 1} + A_{\sigma 2} + A_{\sigma 3} = \frac{\pi}{6}\left[65^{3/2} - 1\right] + 30\pi + 5^2\pi = \boxed{\frac{\pi}{6}\left[65^{3/2} + 329\right]}$

Examen de Fundamentos Matemáticos I
Especialidades Eléctrica y Electrónica 07-09-2005

SOLUCIÓN

A.1.- Demostrar que el lugar geométrico de los puntos del plano complejo para los cuales se cumple

$$\arg\left(\frac{z+2}{z-2}\right)=\frac{\pi}{4}; \quad (z=x+yi),$$

es una circunferencia con centro en $C(0,-2)$ y radio $r=2\sqrt{2}$. [6 PUNTOS]

$$\frac{z+2}{z-2}=\frac{x+2+iy}{x-2+iy}=\frac{[x+2+iy][x-2-iy]}{(x-2)^2+y^2}=\frac{x^2+y^2-4-4yi}{(x-2)^2+y^2} \quad \to$$

$$\arg\left(\frac{z+2}{z-2}\right)=arctg\left(\frac{-4y}{x^2+y^2-4}\right)=\frac{\pi}{4} \quad \to \quad \frac{-4y}{x^2+y^2-4}=tg\left(\frac{\pi}{4}\right)=1 \quad \to$$

$$x^2+y^2+4y-4=0 \quad \Rightarrow \quad x^2+(y+2)^2=\left(2\sqrt{2}\right)^2$$

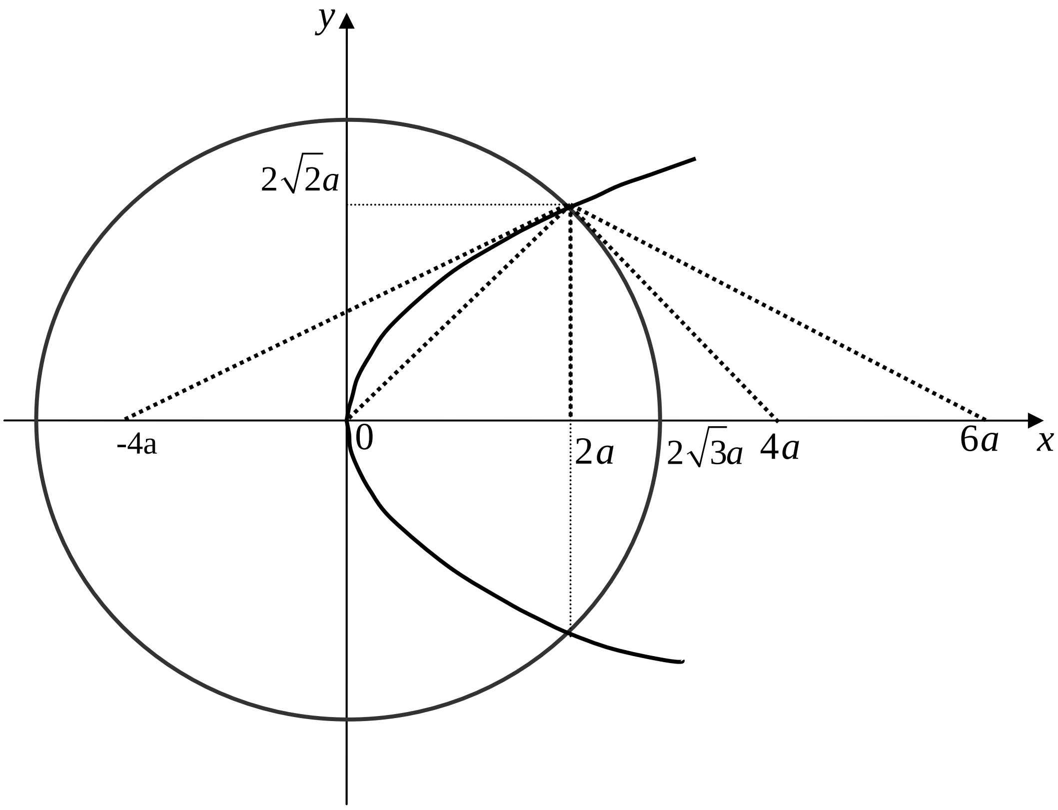
Resulta una circunferencia con centro en $C(0,-2)$ y radio $r=2\sqrt{2}$

A.2.- La circunferencia $x^2+y^2-12a^2=0$ y la parábola $y^2-4ax=0$ se cortan en dos puntos P y Q. Demostrar que las rectas tangentes y normales a ambas curvas en cualquiera de estos puntos interceptan al eje OX en puntos que junto con el pie de la perpendicular a dicho eje, trazada por el punto de corte, forman cinco puntos equidistantes. [7 PUNTOS]

Las curvas se interceptan, como se muestra en el gráfico, según los puntos P y Q

$$\begin{cases} x^2 + y^2 - 12a^2 = 0 \\ y^2 - 4ax = 0 \end{cases} \rightarrow \begin{cases} x^2 + 4ax - 12a^2 = 0 \\ y^2 - 4ax = 0 \end{cases} \rightarrow \begin{cases} x = 2a \\ y = \pm 2\sqrt{2}a \end{cases} \Rightarrow$$

$$P\left(2a, 2\sqrt{2}a\right); \quad Q\left(2a, -2\sqrt{2}a\right)$$



Rectas tangente y normal a la circunferencia en $P\left(2a, 2\sqrt{2}a\right)$ y cortes con OX :

$$x^2 + y^2 - 12a^2 = 0 \xrightarrow{D} 2x + 2yy' = 0 \rightarrow y' = -x/y \rightarrow y'(P) = -1/\sqrt{2}$$

$$\begin{cases} y - y_0 = y'(x - x_0) \rightarrow y - 2\sqrt{2}a = \left(-1/\sqrt{2}\right)(x - 2a) \xrightarrow{y=0} x = 6a \\ y - y_0 = \left(-1/y'\right)(x - x_0) \rightarrow y - 2\sqrt{2}a = \sqrt{2}\,(x - 2a) \xrightarrow{y=0} x = 0 \end{cases}$$

Rectas tangente y normal a la parábola en $P\left(2a, 2\sqrt{2}a\right)$ y cortes con OX :

$$y^2 - 4ax = 0 \xrightarrow{D} 2yy' - 4a = 0 \rightarrow y' = 2a/y \rightarrow y'(P) = 1/\sqrt{2}$$

$$\begin{cases} y - y_0 = y'(x - x_0) & \rightarrow \quad y - 2\sqrt{2}a = \left(1/\sqrt{2}\right)(x - 2a) \xrightarrow{y=0} x = -2a \\ y - y_0 = \left(-1/y'\right)(x - x_0) & \rightarrow \quad y - 2\sqrt{2}a = -\sqrt{2}\,(x - 2a) \xrightarrow{y=0} x = 4a \end{cases}$$

En efecto, se observa que los cinco puntos considerados equidistan (2a) unidades.

A.3.- Hallar la derivada de la función $z = 3x^4 - xy + y^3$ en el punto $M(1,2)$ según la dirección de la bisectriz del primer cuadrante. [6 PUNTOS]

La derivada direccional se define según; $\dfrac{\partial z}{\partial e} = \overrightarrow{\nabla z}.\overrightarrow{V_u}$

Donde $\overrightarrow{\nabla z}$ designa el gradiente de z en el punto y $\overrightarrow{V_u}$ un vector unitario según la dirección de derivación. En nuestro caso:

$$\overrightarrow{\nabla z} = \left[12x^3 - y, -x + 3y^2\right] \quad \rightarrow \quad \overrightarrow{\nabla z}(1,2) = [10,11]$$

El vector unitario en la dirección $\varphi = \pi/4$ es;

$$\overrightarrow{V_u} = \left[\cos(\pi/4), \operatorname{sen}(\pi/4)\right] = \left[1/\sqrt{2}, 1/\sqrt{2}\right] = \left(1/\sqrt{2}\right)[1,1]$$

Se obtiene; $\overrightarrow{\nabla z}.\overrightarrow{V_u} = \overrightarrow{V_u} = [10,11].\left[1/\sqrt{2}, 1/\sqrt{2}\right] = \dfrac{10+11}{\sqrt{2}} = \dfrac{21}{\sqrt{2}}$

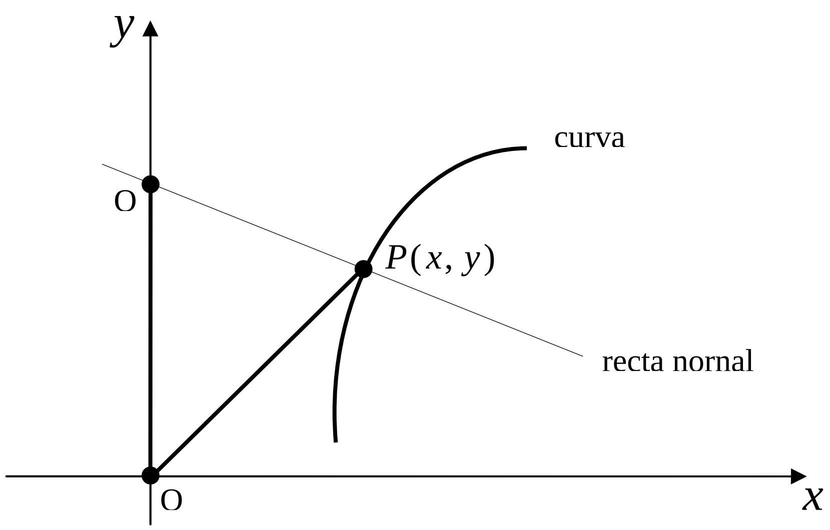
B.1.- Calcular el valor de la expresión $E \equiv y\dfrac{\partial^2 z}{\partial x^2} + x\dfrac{\partial^2 z}{\partial x \partial y}$ para la función $z = y.\varphi\left(x^2 - y^2\right)$, donde φ designa una función arbitraria derivable. [6 PUNTOS]

$$z = y.\varphi\left(x^2 - y^2\right) \quad \rightarrow \quad \begin{cases} \dfrac{\partial z}{\partial x} = y\varphi'(2x) = 2xy\varphi' \\ \dfrac{\partial z}{\partial y} = \varphi + y\varphi'(-2y) = \varphi - 2y^2\varphi' \end{cases} \quad \rightarrow$$

$$\rightarrow \quad \begin{cases} \dfrac{\partial^2 z}{\partial x^2} = 2y\varphi' + 2xy\varphi''(2x) = 2y\varphi' + 4x^2y\varphi'' \\ \dfrac{\partial^2 z}{\partial x \partial y} = 2x\varphi' + 2xy\varphi''(-2y) = 2x\varphi' - 4xy^2\varphi'' \end{cases} \quad \rightarrow$$

$$y\frac{\partial^2 z}{\partial x^2} + x\frac{\partial^2 z}{\partial x \partial y} = y\left(2y\varphi' + 4x^2y\varphi''\right) + x\left(2x\varphi' - 4xy^2\varphi''\right) = 2\left(x^2 + y^2\right)\varphi'$$

B.2.- Se consideran las curvas del plano para las cuales se cumple que el segmento interceptado sobre el eje de ordenadas por la normal trazada en un punto cualquiera $P(x,y)$ es igual a la distancia entre dicho punto y el origen de coordenadas. Probar que la ecuación diferencial del haz de curvas es $xdx+\left(y-\sqrt{x^2+y^2}\right)dy=0$, y determinar sus **trayectorias ortogonales**. [6 PUNTOS]



Ecuación diferencial del haz de curvas

Intersección de la recta normal con el eje $P(x,y)$:

$$Y-y=y'(X-x)\ \xrightarrow{X=0}\ Y\equiv OQ=y+x/y';\quad OP=\sqrt{x^2+y^2}\quad\Rightarrow$$

$$OQ=OP\quad\to\quad y+x/y'=\sqrt{x^2+y^2}\quad\Rightarrow\quad xdx+\left(y-\sqrt{x^2+y^2}\right)dy=0$$

Trayectorias ortogonales

En la ecuación diferencial de las curvas se cambia (y') por $(-1/y')$:

$$y+x/y'=\sqrt{x^2+y^2}\quad\xrightarrow{y'\equiv -1/y'}\quad y'=\frac{y-\sqrt{x^2+y^2}}{x}$$

Ecuación diferencial homogénea que se integra haciendo,

$$y = xu \xrightarrow{D} y' = u + xu' \xrightarrow{ED} u + xu' = u - \sqrt{1+u^2} \rightarrow \frac{du}{\sqrt{1+u^2}} + \frac{dx}{x} = 0$$

$$\rightarrow \ln\left|u + \sqrt{1+u^2}\right| + \ln|x| = C \rightarrow \ln\left|\left[(y/x) + \sqrt{1+(y/x)^2}\right]x\right| = C$$

$$\rightarrow y + \sqrt{x^2+y^2} = e^C \Rightarrow y + \sqrt{x^2+y^2} = A, \quad (A \neq 0)$$

B.3.- Obtener, **a elección**, una de las coordenadas $x(t)$ e $y(t)$, solución del sistema de ecuaciones diferenciales con condiciones iniciales:

$$\begin{cases} x''(t) + y(t) = 4 \\ y''(t) + x(t) = 0 \\ x(0) = x'(0) = y(0) = y'(0) = 0 \end{cases}$$

[7 PUNTOS]

Al aplicar el operador de Laplace e introducir las condiciones iniciales:

$$\begin{cases} x''(t) + y(t) = 4 \\ y''(t) + x(t) = 0 \\ x(0) = x'(0) = y(0) = y'(0) = 0 \end{cases} \rightarrow \begin{cases} p^2 X(p) + Y(p) = 4/p \\ X(p) + p^2 Y(p) = 0 \end{cases}$$

Para resolver en $X(p)$ e $Y(p)$ se aplica la regla de Cramer.

$$X(p) = \frac{\begin{vmatrix} 4/p & 1 \\ 0 & p^2 \end{vmatrix}}{\begin{vmatrix} p^2 & 1 \\ 1 & p^2 \end{vmatrix}} = \frac{4p}{p^4 - 1} = \frac{4p}{(p+1)(p-1)(p^2+1)}$$

$$Y(p) = \frac{\begin{vmatrix} p^2 & 4/p \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} p^2 & 1 \\ 1 & p^2 \end{vmatrix}} = \frac{-4}{p(p^4-1)} = \frac{-4}{p(p+1)(p-1)(p^2+1)}$$

Resulta más sencillo resolver en $x(t)$. Se descompone $X(p)$ en fracciones simples y se aplica el operador inverso de Laplace:

$$X(p) = \frac{4p}{(p+1)(p-1)(p^2+1)} \equiv \frac{A}{p+1} + \frac{B}{p-1} + \frac{Cp+D}{p^2+1} =$$

$$\frac{A(p-1)\left(p^2+1\right)+B(p+1)\left(p^2+1\right)+(Cp+D)\left(p^2-1\right)}{(p+1)(p-1)\left(p^2+1\right)} \to \begin{cases} p=1 \to 4=4B \to B=1 \\ p=-1 \to -4=-4A \to A=1 \\ p^3 : 0=A+B+C \to C=-2 \\ p^0 : 0=-A+B-D \to D=0 \end{cases}$$

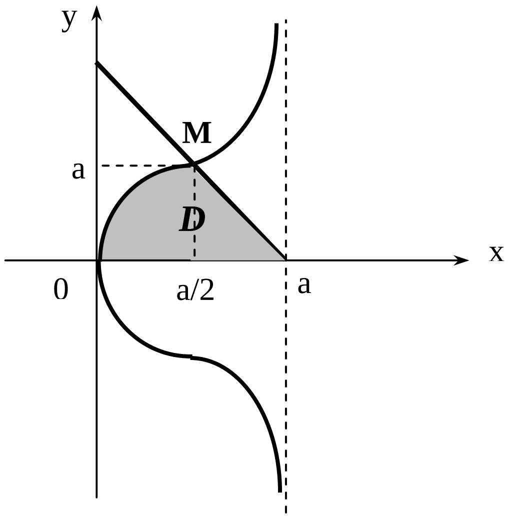
$$X(p)=\frac{1}{p+1}+\frac{1}{p-1}-\frac{2p}{p^2+1} \xrightarrow{L^{-1}} x(t)=e^{-t}+e^{t}-2\cos t=2(cht-\cos t)$$

C.1.-Demostrar que el área D que limitan en el primer cuadrante las líneas,

$$y^2=\frac{a^2x}{a-x}; \quad 2x+y=2a; \quad y=0, \quad \text{es } A=\frac{a^2(\pi-1)}{4}$$

Probar que si D gira alrededor de OX genera un cuerpo de volumen: $V=\dfrac{3\ln 2-1}{3}\pi a^3$

Indicación: Las líneas se cortan en el punto $M\left(a/2, a\right)$ [11 PUNTOS]



1.- Cálculo del área del dominio D

$$A_D=\iint_D dxdy=\int_0^a dy\int_{ay^2/(a^2+y^2)}^{(2a-y)/2}dx=\int_0^a\left[\frac{(2a-y)}{2}-\frac{ay^2}{a^2+y^2}\right]dy=\frac{1}{2}\left[\frac{(2a-y)^2}{-2}\right]\Bigg|_0^a+$$

$$+\int_0^a\left[-a+\frac{a^3}{a^2+y^2}\right]dy=\frac{-1}{4}\left[a^2-4a^2\right]+\left[-ay+\frac{a^3}{a}arctg\frac{x}{a}\right]\Bigg|_0^a=$$

$$A_D=\frac{3a^2}{4}-a^2+a^2\cdot arctg1=\frac{-a^2}{4}+\frac{a^2\pi}{4}=\frac{a^2}{4}(\pi-1)$$

2.- Cálculo del Volumen del cuerpo de revolución alrededor de OX

$$V_C=\pi\int_0^{a/2}y^2dx+\pi\int_{a/2}^{a}y^2dx=\pi\int_0^{a/2}\frac{a^2x}{a-x}dx+\pi\int_{a/2}^{a}\left[2(a-x)\right]^2dx=$$

$$V_C = \pi\int_0^{a/2}\left[-a^2+\frac{a^3}{a-x}\right]dx + 4\pi\int_{a/2}^{a}(a-x)^2\,dx = \pi\left[-a^2x - a^3L_n\left|a-x\right|\right]\Big|_0^{a/2} +$$

$$+4\pi\left[\frac{(a-x)^3}{-3}\right]\Bigg|_{a/2}^{a} = \pi\left[\frac{-a^3}{2} - a^3L_n\frac{a}{2} + a^3L_na\right] + \frac{4\pi}{3}\left[0+\frac{a^3}{8}\right] =$$

finalmente:

$$V_C = \pi\left[\frac{-a^3}{2}+\frac{a^3}{6} - a^3L_na + a^3L_n2 + a^3L_na\right] = \pi\left[\frac{-a^3}{3} + a^3L_n2\right] = \pi a^3\left[\frac{3L_n2-1}{3}\right]$$

C.2.- Un cuerpo $[V]$ situado en el semiespacio $y \geq 0$ está limitado por las superficies:

$$[\sigma_c]:\quad x^2+z^2=9\,;\qquad [\sigma_p]:\quad x^2+9\left(z^2-y\right)=0\qquad [\sigma_3]:\quad y=0.$$

---

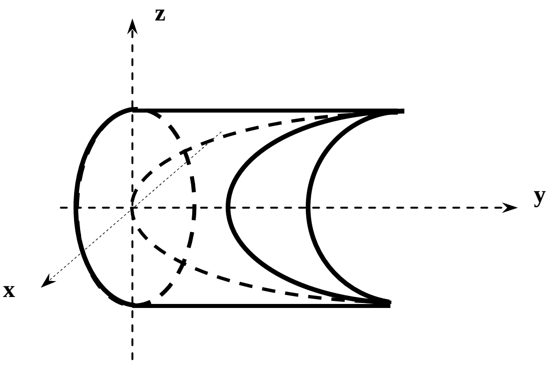
1. Mostrar que su volumen es $V=\dfrac{45\pi}{2}$.

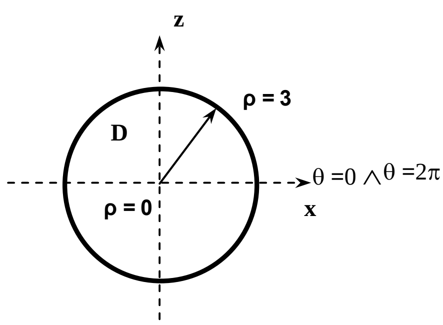
2. Probar que $\iiint_V y\,dxdydz = 3\int_0^{\pi/2}\left(\cos^4\theta + 18\,\text{sen}^2\theta\cos^2\theta + 81\,\text{sen}^4\theta\right)dz$

3. Establecer la posición del centro de gravedad de $[V]$ [11 PUNTOS]

Sugerencia: Puede aplicarse la propiedad de las integrales eulerianas:

$$\int_0^{\pi/2}\text{sen}^{(2m-1)}t\cos^{(2n-1)}t\,dt = \frac{B(m,n)}{\mathbf{z}\ 2}$$





Proyección cilíndrica

1.- Cálculo del Volumen del cuerpo $[V]$

$$V_C = \iiint_C dxdydz = \iiint_C \rho\,d\rho\,d\theta\,dy = \iint_D \rho\,d\rho\,d\theta\int_0^{\frac{x^2+9z^2}{9}}dy = \frac{1}{9}\cdot\iint_D (x^2+9z^2)\rho\,d\rho\,d\theta =$$

$$V_C = \frac{1}{9}\cdot\iint_D(\cos^2\theta + 9\,\text{sen}^2\theta)\rho^3\,d\rho\,d\theta = \frac{1}{9}\cdot\int_0^{2\pi}(1+8\,\text{sen}^2\theta)d\theta\int_0^3\rho^3\,d\rho =$$

$$V_C = \frac{1}{9}\cdot\int_0^{2\pi}\left[1+\frac{8}{2}(1-\cos 2\theta)\right]d\theta\cdot\left[\frac{\rho^4}{4}\right]\Bigg|_0^3 = \frac{81}{9\cdot 4}\cdot\left[5\theta - 2\,\text{sen}\,2\theta\right]\Big|_0^{2\pi} = \frac{45}{2}\pi \quad (u^3)$$

---

2. Probar que: $\iiint_V y dxdydz = 3\int_0^{\pi/2}\left(\cos^4\theta + 18\,\text{sen}^2\theta\cos^2\theta + 81\,\text{sen}^4\theta\right)dz$

$$M_{XOZ} = \iiint_V y dxdydz = \iint_D dx\,dz\int_0^{\frac{x^2+9z^2}{9}} y dy = \frac{1}{9^2\cdot 2}\cdot\iint_D\left[(x^2+9z^2)^2\right]dx\,dz =$$

$$M_{XOZ} = \frac{1}{162}\iint_D(\cos^2\theta + 9\,\text{sen}^2\theta)^2\rho^4\,\rho\,d\rho\,d\theta = \frac{1}{162}\int_0^{2\pi}(\cos^2\theta+9\,\text{sen}^2\theta)^2 d\theta\int_0^3\rho^5\,d\rho =$$

$$M_{XOZ} = \frac{1}{162}\int_0^{2\pi}(\cos^2\theta+9\,\text{sen}^2\theta)^2 d\theta\cdot\left[\frac{\rho^6}{6}\right]\Bigg|_0^3 = \frac{81\cdot 9}{162\cdot 6}\int_0^{2\pi}(\cos^2\theta+9\,\text{sen}^2\theta)^2 d\theta =$$

$$M_{XOZ} = \frac{3}{4}\cdot 4\int_0^{\pi/2}\left(\cos^4\theta + 18\,\text{sen}^2\theta\cos^2\theta + 81\,\text{sen}^4\theta\right)d\theta =$$

$$M_{XOZ} = 3\int_0^{\pi/2}\left(\cos^4\theta + 18\,\text{sen}^2\theta\cos^2\theta + 81\,\text{sen}^4\theta\right)d\theta$$

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3. **Centro de gravedad.- Cálculo**

$$M_{XOZ} = 3\int_0^{\pi/2}\cos^4\theta\,d\theta + 3\cdot 18\int_0^{\pi/2}\text{sen}^2\theta\cos^2\theta\,d\theta + 3\cdot 81\int_0^{\pi/2}\text{sen}^4\theta\,d\theta = 3\,I_1 + 54\,I_2 + 243\,I_3$$

$$M_{XOZ} = 3\cdot\frac{1}{2}B(5/2,1/2) + 54\cdot\frac{1}{2}B(3/2,3/2) + 243\cdot\frac{1}{2}B(1/2,5/2) =$$

$$M_{XOZ} = \frac{3}{2}\frac{\Gamma(5/2)\cdot\Gamma(1/2)}{\Gamma(3)} + \frac{54}{2}\frac{\Gamma(3/2)\cdot\Gamma(3/2)}{\Gamma(3)} + \frac{243}{2}\frac{\Gamma(1/2)\cdot\Gamma(5/2)}{\Gamma(3)} =$$

$$M_{XOZ} = \frac{3}{2}\frac{3/2\cdot 1/2\left(\sqrt{\pi}\right)^2}{2} + 27\frac{1/2\cdot 1/2\left(\sqrt{\pi}\right)^2}{2} + \frac{243}{2}\frac{3/2\cdot 1/2\left(\sqrt{\pi}\right)^2}{2} =$$

$$M_{XOZ} = \frac{9}{16}\pi + \frac{27}{8}\pi + \frac{729}{16}\pi = \frac{792}{16}\pi = \frac{99}{2}\pi$$

Por la simetría del cuerpo $[V]$ alrededor del eje OY, se sabe que su centro de gravedad estará situado en el antedicho eje OY. Por tanto:

$$y_G = \frac{M_{XOZ}}{V_V} = \frac{99\pi/2}{45\pi/2} = \frac{99}{45} = \frac{11}{5} \qquad ; \qquad G_V = \left(0,\frac{11}{5},0\right)$$

Examen de Fundamentos Matemáticos I
Especialidades Eléctrica y Electrónica 23-01-2006

PRIMER CUATRIMESTRAL

A1. Justificar que, $4x-4y-21=0,$ **es una recta tangente a la parábola** $y=x^2-2x-3,$ **paralela a la cuerda que une los puntos** $A(0,-3);\ B(3,0)$**.** **[4 puntos]**

La función es continua y derivable en todo el eje. Si se aplica el teorema de Lagrange en el intervalo $[0,3]$:

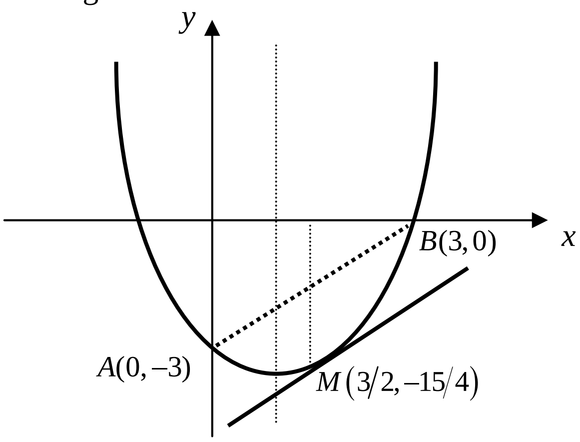
$$y=x^2-2x-3 \quad\to\quad y'=2x-2 \quad \xrightarrow{T.Lagrange}$$

$$\frac{f(b)-f(a)}{b-a}=f'(c),\quad c\in(a,b):\ \frac{0-(-3)}{3-0}=1=2c-2\ \to\ c=3/2 \quad\Rightarrow\quad y(3/2)=-15/4$$

Luego en $M\left(3/2,-15/4\right)$ la tangente es paralela a la cuerda $\overline{AB}$. Su ecuación es:

$$y-y(c)=y'(c)(x-c) \quad\to\quad y+15/4=1\left(x-3/2\right) \quad\Rightarrow\quad 4x-4y-21=0$$

El ejercicio se ilustra con la figura.



A2. Comprobar que $y=\left(x+\sqrt{x^2+1}\right)^3$ **satisface la ecuación diferencial**

$$\left(1+x^2\right)y''+xy'-9y=0,$$

y demostrar que sus derivadas cumplen la relación recurrente:

$$\left(1+x^2\right)y^{(n+2)}+(2n+1)xy^{(n+1)}+\left(n^2-9\right)y^{(n)}=0.$$ **[8 puntos]**

Derivando una vez:

$$y=\left(x+\sqrt{x^2+1}\right)^3 \xrightarrow{D} y'=3\left(x+\sqrt{x^2+1}\right)^2\left(1+\frac{x}{\sqrt{x^2+1}}\right)=\frac{3\left(x+\sqrt{x^2+1}\right)^3}{\sqrt{x^2+1}}$$

Al quitar denominadores y derivar de nuevo:

$$y'\sqrt{x^2+1}=3\left(x+\sqrt{x^2+1}\right)^3 \xrightarrow{D} y''\sqrt{x^2+1}+\frac{xy'}{\sqrt{x^2+1}}=9\left(x+\sqrt{x^2+1}\right)^2\left(1+\frac{x}{\sqrt{x^2+1}}\right)\rightarrow$$

$$\left(1+x^2\right)y''+xy'=9\left(x+\sqrt{x^2+1}\right)^3 \quad\Rightarrow\quad \left(1+x^2\right)y''+xy'-9y=0$$

Para obtener la relación de recurrencia se aplica la fórmula de Leibniz a los miembros de la ecuación diferencial.

$$\binom{n}{0}y^{(n+2)}\left(1+x^2\right)+\binom{n}{1}y^{(n+1)}2x+\binom{n}{2}y^{(n)}2+\binom{n}{0}y^{(n+1)}x+\binom{n}{1}y^{(n)}-9y^{(n)}=0 \quad\rightarrow$$

$$\left(1+x^2\right)y^{(n+2)}+2nxy^{(n+1)}+n(n-1)y^{(n)}+xy^{(n+1)}+ny^{(n)}-9y^{(n)}=0 \quad\Rightarrow$$

$$\left(1+x^2\right)y^{(n+2)}+(2n+1)xy^{(n+1)}+\left(n^2-9\right)y^{(n)}=0$$

A3. Hallar las cinco primeras derivadas de la función en $x=0$ y justificar la estimación

$$y\approx 1+3x+\frac{9}{2}x^2+4x^3+\frac{15}{8}x^4 .$$ [8 puntos]

Se concreta la ley de recurrencia para $x=0$:

$$\left(1+x^2\right)y^{(n+2)}+(2n+1)xy^{(n+1)}+\left(n^2-9\right)y^{(n)}=0 \xrightarrow{x=0} y^{(n+2)}(0)=\left(9-n^2\right)y^{(n)}(0)$$

En $x=0$: $y(0)=1$; $y'(0)=3$. Dando valores $n=0,1,2\,y\,3$, se calculan las siguientes derivadas:

$$y^{(n+2)}(0)=\left(9-n^2\right)y^{(n)}(0) \quad\rightarrow\quad \begin{cases} n=0: \quad y''(0)=9y(0)=9 \\ n=1: \quad y'''(0)=8y'(0)=24 \\ n=2: \quad y^{IV}(0)=5y''(0)=45 \\ n=3: \quad y^{V}(0)=0y'''(0)=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Para justificar la estimación se desarrolla la función en potencias de (x) mediante la fórmula de MacLaurin:

$$y=y(0)+\frac{x}{1!}y'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y'''(0)+\frac{x^4}{4!}y^{IV}(0)+\frac{x^5}{5!}y^{V}(0)+\ldots \quad\rightarrow$$

$$y=1+3x+\frac{9}{2}x^2+4x^3+\frac{15}{8}x^4+R_5 \quad\Rightarrow\quad y\approx 1+3x+\frac{9}{2}x^2+4x^3+\frac{15}{8}x^4$$

B1. Probar que la función $z=z(x,y)$ **definida de forma implícita según** $z^2=x^2.\phi(z/y)$, **donde** ϕ **designa una función arbitraria diferenciable, satisface a la ecuación de Euler para funciones homogéneas**

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z\,\textbf{.}$$ **[10 puntos]**

Cálculo de las derivadas parciales empleando la notación $\dfrac{\partial z}{\partial x}\equiv p;\quad \dfrac{\partial z}{\partial y}\equiv p:$

$$z^2=x^2.\phi(z/y)\quad\rightarrow\quad\begin{cases}\xrightarrow{\partial/\partial x}\quad 2zp=2x.\phi+x^2\phi'.\left(p/y\right)\quad\rightarrow\quad p=\dfrac{2xy\phi}{2yz-x^2\phi'}\\[2ex] \xrightarrow{\partial/\partial y}\quad 2zq=x^2\phi'\left(\dfrac{yq-z}{y^2}\right)\quad\rightarrow\quad q=\dfrac{-x^2z\phi'}{2y^2z-x^2y\phi'}\end{cases}$$

Al sustituir en la ecuación en derivadas parciales:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\frac{2xy\phi}{2yz-x^2\phi'}+y\frac{-x^2z\phi'}{2y^2z-x^2y\phi'}=\frac{2x^2y\phi-x^2z\phi'}{2yz-x^2\phi'}\equiv$$
$$\equiv\frac{2yz^2-x^2z\phi'}{2yz-x^2\phi'}=\frac{2yz-x^2\phi'}{2yz-x^2\phi'}z=z$$

B2. Determinar los extremos relativos de la función $z=xy+16/x+4/y$. **[10 puntos]**

Puntos críticos: Se obtiene como único punto $M(4,1)$.

$$z=xy+16/x+4/y\quad\rightarrow\quad\begin{cases}\dfrac{\partial z}{\partial x}=0\rightarrow y-16/x^2=0\rightarrow x^2y-16=0\\[2ex] \dfrac{\partial z}{\partial y}=0\rightarrow x-4/y^2=0\rightarrow xy^2-4=0\end{cases}\quad\rightarrow\quad x=4;\,y=1$$

A través del criterio del hessiano:

$$H\left[z(x,y)\right]=\begin{vmatrix}\dfrac{\partial^2 z}{\partial x^2} & \dfrac{\partial^2 z}{\partial x\partial y}\\[2ex] \dfrac{\partial^2 z}{\partial y\partial x} & \dfrac{\partial^2 z}{\partial y^2}\end{vmatrix}=\begin{vmatrix}\dfrac{32}{x^3} & 1\\[2ex] 1 & \dfrac{8}{y^3}\end{vmatrix}=\frac{256}{x^3y^3}-1\quad\rightarrow$$

$$H\left[z(4,1)\right]=3>0;\quad\frac{\partial^2 z}{\partial x^2}(4,1)=\frac{1}{2}>0\quad\Rightarrow\quad z(4,1)=12\ \text{ es un } \textbf{mínimo}$$

C.1 Mediante sustituciones adecuadas reducir a eulerianas y verificar los resultados que se indican para <u>dos de las tres</u> integrales siguientes: [10 puntos]

$$I_1 = \int_0^\infty \frac{dx}{\left(x^2+4\right)^3} = \frac{3\pi}{512} \quad ; \quad I_2 = \int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} = \frac{\Gamma\left(1-1/n\right)\Gamma\left(1/n\right)}{n}$$

$$I_3 = \int_0^1 x\ln^5\left(1/x\right)dx = \frac{15}{8}$$

$$I_1 = \int_0^\infty \frac{dx}{\left(x^2+4\right)^3} \to \left| \begin{array}{l} x = 2\,tg\mathrm{t} \\ dx = \dfrac{2}{\cos^2 t}\ dt \end{array} \right| \to I_1 = \int_0^{\pi/2} \frac{1}{\left(4tg^2\mathrm{t}+4\right)^3}\frac{2}{\cos^2 t}\ \mathrm{dt} =$$

$$I_1 = \frac{2}{4^3}\int_0^{\pi/2}\cos^4 t\ \ d\mathrm{t} = \frac{1}{32}\cdot\frac{1}{2}\cdot\beta\left(\frac{5}{2},\frac{1}{2}\right) = \frac{1}{64}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3\right)} =$$

$$I_1 = \frac{1}{64}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2} = \frac{3\pi}{512}$$

$$I_2 = \int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} \to \left| \begin{array}{l} x^n = \mathrm{t} \\ x = \mathrm{t}^{1/n} \\ \dfrac{dx}{n} = \dfrac{1}{n}\mathrm{t}^{(1/n)-1}\ dt \end{array} \right| \to I_2 = \frac{1}{n}\int_0^1 (1-t)^{-1/n}t^{(1/n)-1}\ d\mathrm{t} = \frac{1}{n}\cdot\beta\left(1-\frac{1}{n},\frac{1}{n}\right)$$

por tanto: $$I_2 = \frac{1}{n}\cdot\frac{\Gamma\left(1-1/n\right)\Gamma\left(1/n\right)}{\Gamma\left(1\right)} = \frac{\Gamma\left(1-1/n\right)\Gamma\left(1/n\right)}{n}$$

$$I_3 = \int_0^1 x\ln^5\left(1/x\right)dx \to \left| \begin{array}{l} x = e^{-t} \\ dx = -e^{-t}dt \end{array} \right| \to I_3 = -\int_\infty^0 e^{-t}t^5e^{-t}dt = \int_0^\infty e^{-2t}t^5\,dt =$$

$$I_3 = \int_0^\infty e^{-2t}t^5\,d\mathrm{t} \to \left| \begin{array}{l} 2t = z \\ dt = \dfrac{1}{2}dz \end{array} \right| \to I_3 = \frac{1}{2}\int_0^\infty e^{-z}\left(\frac{z}{2}\right)^5 dz = \frac{1}{2^6}\int_0^\infty z^5e^{-z}\,dz = \frac{1}{2^6}\Gamma(6)$$

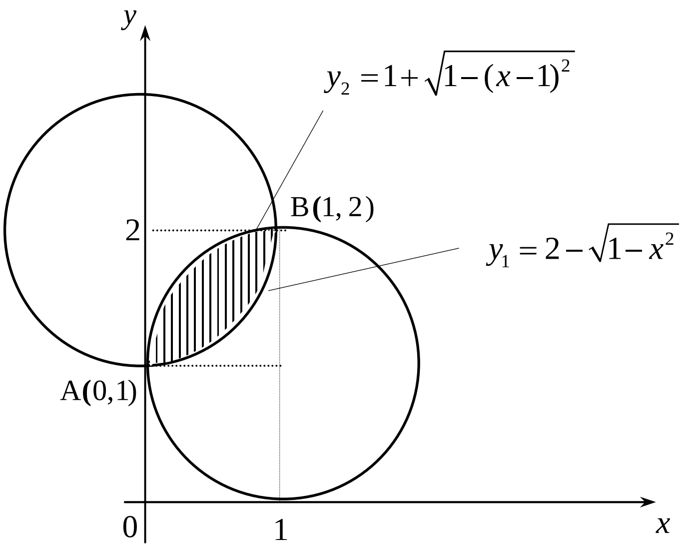
$$I_3 = \frac{1}{2^6}\Gamma(6) = \frac{1}{2^6}\cdot 5! = \frac{5\cdot4\cdot3\cdot2}{2^6} = \frac{15}{8}$$

C.2 Evaluar mediante cálculo integral el área plana definida mediante:

$[L_1]\quad x^2+y^2-2x-2y+1\le 0;\quad [L_2]\quad x^2+y^2-4y+3\le 0$ **[10 puntos]**

La frontera del dominio está formada por dos circunferencias:

$$\begin{cases} L_1: \quad (x-1)^2+(y-1)^2=1 \\ L_2: \quad x^2+(y-2)^2=1 \end{cases} \to \begin{cases} centro:(1,1);\ \mathrm{R}=1 \\ centro:(0,2);\ \mathrm{R}=1 \end{cases} \to \begin{cases} y=1+\sqrt{1-(x-1)^2} \\ y=2-\sqrt{1-x^2} \end{cases}$$



Cálculo del área del dominio

$$A=\int_0^1\left[1+\sqrt{1-(x-1)^2}-(2-\sqrt{1-x^2})\right]dx=\int_0^1\left[\sqrt{1-(x-1)^2}+\sqrt{1-x^2}-1\right]dx$$

$$A=\int_0^1\sqrt{1-(x-1)^2}\ dx+\int_0^1\sqrt{1-x^2}\ dx-\int_0^1 dx=I_1+I_2-1$$

$$I_1=\int_0^1\sqrt{1-(x-1)^2}\ dx \to \left| \begin{array}{l} x-1=\sin t \to dx=\cos t\,dt \\ \begin{cases} x=1\to t=0 \\ x=0\to t=-\pi/2 \end{cases} \end{array} \right| \to$$

$$I_1=\int_{-\pi/2}^0 \cos^2 t\,dt=\left|\frac{t}{2}+\frac{\mathrm{sen}\,2t}{4}\right|_{-\pi/2}^0=\frac{\pi}{4}$$

$$I_2=\int_0^1\sqrt{1-x^2}\ dx \to \left| \begin{array}{l} x=\sin t \to dx=\cos t\,dt \\ \begin{cases} x=1\to t=\dfrac{\pi}{2} \\ x=0\to t=0 \end{cases} \end{array} \right| \to I_2=\int_0^{\pi/2}\cos^2 t\,dt=\frac{\pi}{4}$$

Finalmente: $A=I_1+I_2-1=2\dfrac{\pi}{4}-1=\dfrac{\pi}{2}-1$